1 Waves on Strings

1.1 What is a "wave"?

Difficult to define precisely: here are two "definitions".

• **COULSON (1941)**: "We are all familiar with the idea of a wave; thus, when a pebble is dropped into a pond, water waves travel radially outwards; when a piano is played, the wires vibrate and sound waves spread throughout the room; when a radio station is transmitting, electric waves move through the ether. These are all examples of wave motion, and they have two important properties in common: firstly, **energy is propagated** to distant points; and secondly, the **disturbance travels** through the medium **without** giving the medium as a whole **any permanent displacement**."

• **WHITHAM (1974)**: "...but to cover the whole range of wave phenomena it seems preferable to be guided by the intuitive view that **a wave is any recognizable signal** that is **transferred** from one part of the medium to another **with a recognizable velocity of propagation**."

We begin with, perhaps, the simplest possible example.

1.2 Derivation of Governing PDE

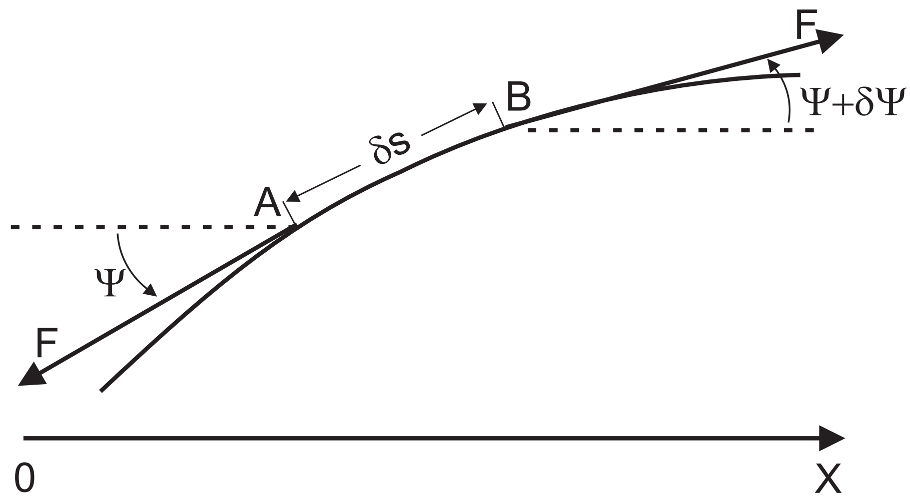

Figure 1: A piece S of a string

• We suppose the string is under tension F, and that its mass per unit length is ρ. We consider transverse motion only ($\perp\ Ox$), and let the displacement be $y(x,t)$; we shall suppose y is small or -more precisely- we suppose $|\ \partial y/\partial x\ | \ll 1$ everywhere.

• Longitudinal motion negligible $\Rightarrow$ F is independent of x (see part ii below). We also take ρ independent of x.

• Apply N2 to a small element of the string AB of length δs.

$$\rho\delta s\frac{\partial^2 y}{\partial t^2} = F\{sin(\psi+\delta\psi) - sin\psi\}. \qquad (1)$$

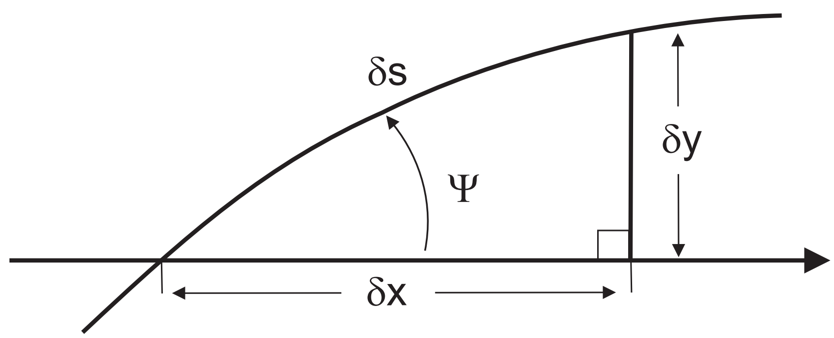

Figure 2: Local geometry of string S

Now, from sketch Fig. 2

$$\delta s^2 \approx \delta x^2 + \delta y^2 \Rightarrow \delta s \approx \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{1/2} \delta x \qquad (2)$$

Therefore, because $|\ \partial y/\partial x\ | \ll 1\ \forall\ x$ (by assumption),

$$\delta s \approx \delta x \tag{3}$$

to highest order. Likewise

$$\tan\psi = \partial y/\partial x \ll 1 \Rightarrow \psi \approx \partial y/\partial x,$$

and, in Eq. (1),

$$\begin{aligned}
\sin(\psi + \delta\psi) - \sin\psi &\approx \cos\psi \cdot \delta\psi \\
&\approx \{1 + \tan^2\psi\}^{-1/2}\delta\psi \\
&\approx \delta\psi \\
&\approx \delta(\partial y/\partial x) \\
&\approx (\partial^2 y/\partial x^2)\delta x.
\end{aligned}$$

Thus Eq.(1) becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{\rho}\frac{1}{\delta x}\frac{\partial^2 y}{\partial x^2}\delta x = \frac{F}{\rho}\frac{\partial^2 y}{\partial x^2}. \tag{4}$$

Finally we have

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \tag{5}$$

where the constant c satisfies

$$c^2 = \frac{F}{\rho}. \tag{6}$$

- Eq. (5) is the 1D wave equation and c is the wave speed.

• (i) For the D string of a violin, $F \approx 55$ N, $\rho \approx 1.4\times10^{-3}$ kgm^{-1} $\Rightarrow c \approx 200$ ms^{-1}

• (ii) We have assumed F is uniform. Hooke's Law $\Rightarrow$ change in $F \propto$ change in length. But

$$\begin{aligned}
\text{change in length} &= \delta s - \delta x \\
&\approx \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{1/2} \delta x - \delta x \\
&\approx \left\{1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right\} \delta x \\
&= \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \delta x
\end{aligned}$$

which is second-order in small quantities $\Rightarrow$ the assumption of uniform F is OK.

• (*iii*) The kinetic energy (KE) of an element of length δs is

$$\frac{1}{2}\rho\delta s\left(\frac{\partial y}{\partial t}\right)^2 \approx \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \delta x,$$

which implies that the KE between $x = a$ and $x = b$ $(> a)$ is

$$\mathrm{KE} = T = \frac{1}{2}\rho\int_a^b\left(\frac{\partial y}{\partial t}\right)^2 dx. \tag{7}$$

The potential energy (PE) of an element of length δs is

$$\begin{aligned} F \times \text{increase in length} &= F(\delta s - \delta x) \\ &\approx \frac{1}{2}F\left(\frac{\partial y}{\partial x}\right)^2 \delta x \quad (\text{from}(ii)). \end{aligned}$$

Thus the PE between $x = a$ and $x = b$ $(> a)$ is

$$\mathrm{PE} = V = \frac{1}{2}F\int_a^b\left(\frac{\partial y}{\partial x}\right)^2 dx. \tag{8}$$

NB T, V are second-order in small quantities, i.e. $(\partial y/\partial x)^2$, $(\partial y/\partial t)^2$, whereas the wave equation Eq. (5) itself is first-order.

1.3 D'Alembert's solution and simple applications

• Unusually we can find the general solution of the wave equation Eq. (5). Change variables from (x, t) to (u, v), where

$$u = x - ct, \qquad v = x + ct. \tag{9}$$

Chain rule $\Rightarrow$

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} = y_u + y_v \Rightarrow \\
\frac{\partial^2 y}{\partial x^2} &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(y_u + y_v) = y_{uu} + 2y_{uv} + y_{vv},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = -cy_u + cy_v \Rightarrow \\
\frac{\partial^2 y}{\partial t^2} &= c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)(y_u - y_v) \\
&= c^2(y_{uu} - 2y_{uv} + y_{vv}).
\end{aligned}$$

- Substitute in the wave equation Eq. (5)

$$c^2(y_{uu} + 2y_{uv} + y_{vv}) = c^2(y_{uu} - 2y_{uv} + y_{vv})$$

$\Rightarrow$

$$y_{uv} = \frac{\partial^2 y}{\partial u \partial v} = 0. \qquad (10)$$

Therefore,

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = 0 \Rightarrow \frac{\partial y}{\partial v} = g_\star(v),$$

where $g_\star$ is any function[1] $\Rightarrow$

$$y = \underbrace{\int^v g_\star(s) ds}_{g(v)} + f(u),$$

where f is any function[1]. Thus

$$y = f(u) + g(v),$$

i.e.

$$y = f(x - ct) + g(x + ct). \qquad (11)$$

Eq. (11) is d'Alembert's solution (the general solution) of the wave equation (5), first published in 1747 [J. le Rond d'Alembert (1717-83)].

[1] Of course f, g must be differentiable (except, perhaps, at isolated points)

• The functions f and g in Eq. (11) are determined by the boundary and initial conditions. For the moment we suppose the string is unbounded in both directions, i.e. $-\infty < x < \infty$.

To begin with, suppose that, at $t = 0$,

$$y(x,0) = \Phi(x), \quad \dot{y}(x,0) = 0. \tag{12}$$

Thus the string is initially at rest $\forall\, x$, but has a displacement given by $y = \Phi(x)$.

From (11) and (12) we must have

$$f(x) + g(x) = \Phi(x), \quad -cf'(x) + cg'(x) = 0.$$

where $'$ denotes "derived function". The second gives $f'(x) = g'(x) \Rightarrow f(x) = g(x) + \alpha$, where α is a constant. The first then gives:

$$f(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\alpha, \quad g(x) = \frac{1}{2}\Phi(x) - \frac{1}{2}\alpha.$$

Thus, from Eq. (11):

$$y(x,t) = \frac{1}{2}\Phi(x - ct) + \frac{1}{2}\Phi(x + ct). \tag{13}$$

1.3.1 Examples

The Heaviside function

The Heaviside [O. Heaviside (1850-1925)] function $H(x)$ is defined by

$$H(x) = \begin{cases} 1 & (x \geq 0) \\ 0 & (x < 0) \end{cases} \tag{14}$$

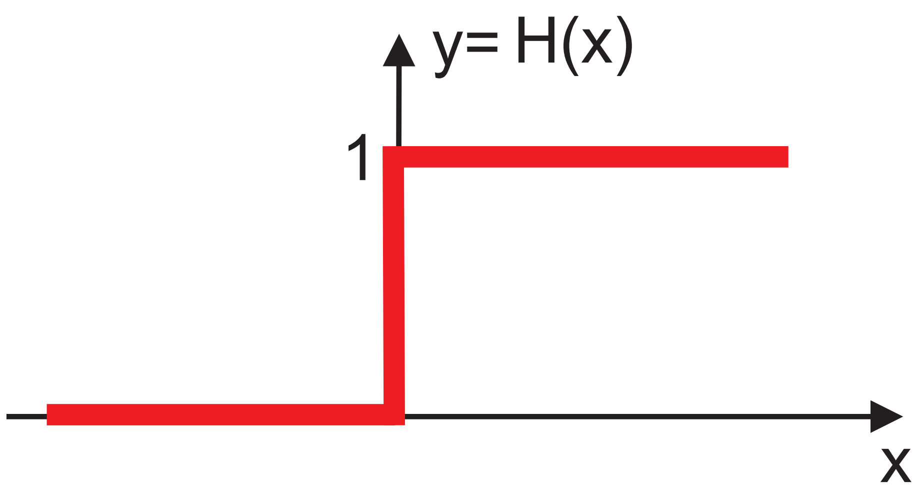

Figure 3: Heaviside function

Example 1

At $t = 0$, an infinite string is at rest and

$$y(x, 0) = b\{H(x + a) - H(x - a)\}, \qquad (15)$$

where $a, b > 0$ constants. Find $y(x, t)$ for $\forall\ x, t$ and sketch your solution.

Solution

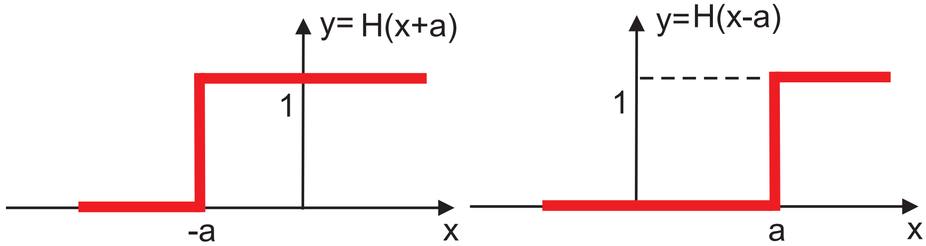

Figure 4: Shifted Heaviside functions

Thus Eq. (15) has the sketch $y(x, 0)$

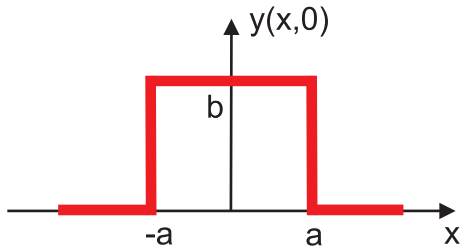

Figure 5: The initial profile $y(x, 0)$

Eq. (13) gives

$$y(x,t) = \frac{b}{2}\{H(x-ct+a) - H(x-ct-a)\} + \frac{b}{2}\{H(x+ct+a) - H(x+ct-a)\} \quad (16)$$

The first term is like $y(x,0)$ except that

- (i) its height is $(1/2)b$, not b, and
- (ii) its end points are $(ct-a, ct+a)$, not $(-a, a)$.

This is a signal with graph like Fig. 5 except for (i) and (ii). Thus the first term in Eq. (16) has graph:

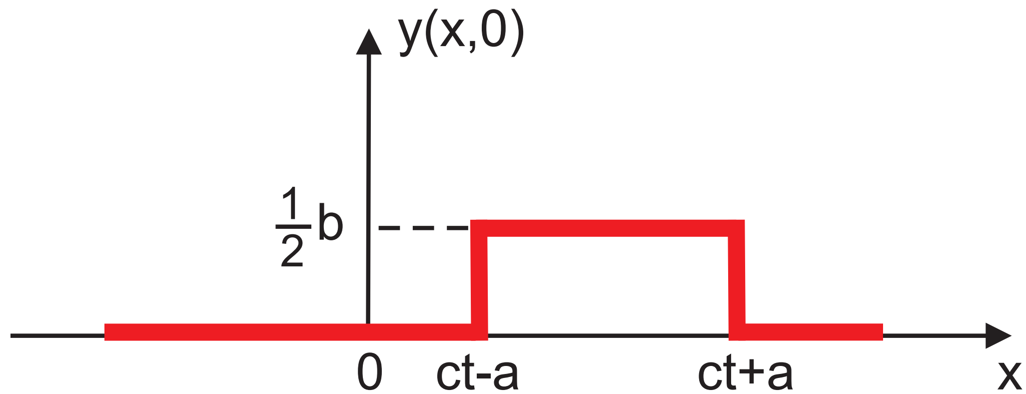

Figure 6: Travelling to right with speed c

Likewise the second term has graph:

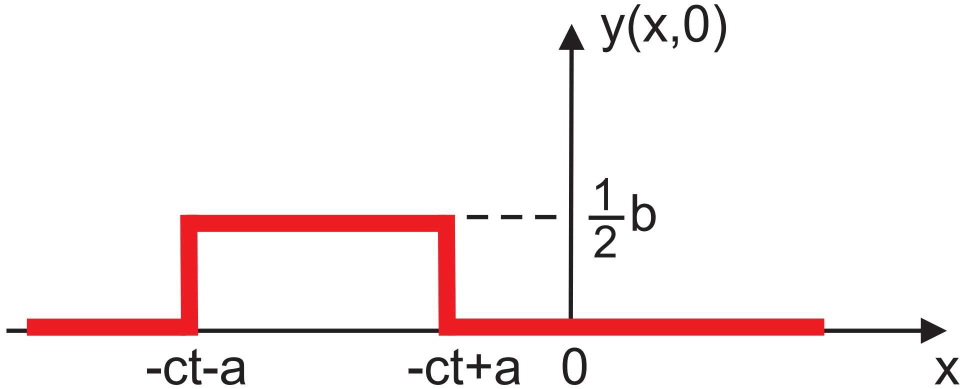

Figure 7: Travelling to left with speed c

The sum of the two pulses has a graph which depends on whether they overlap; this happens for t such that

$$-ct + a > ct - a$$

$\Rightarrow$

$$t < a/c.$$

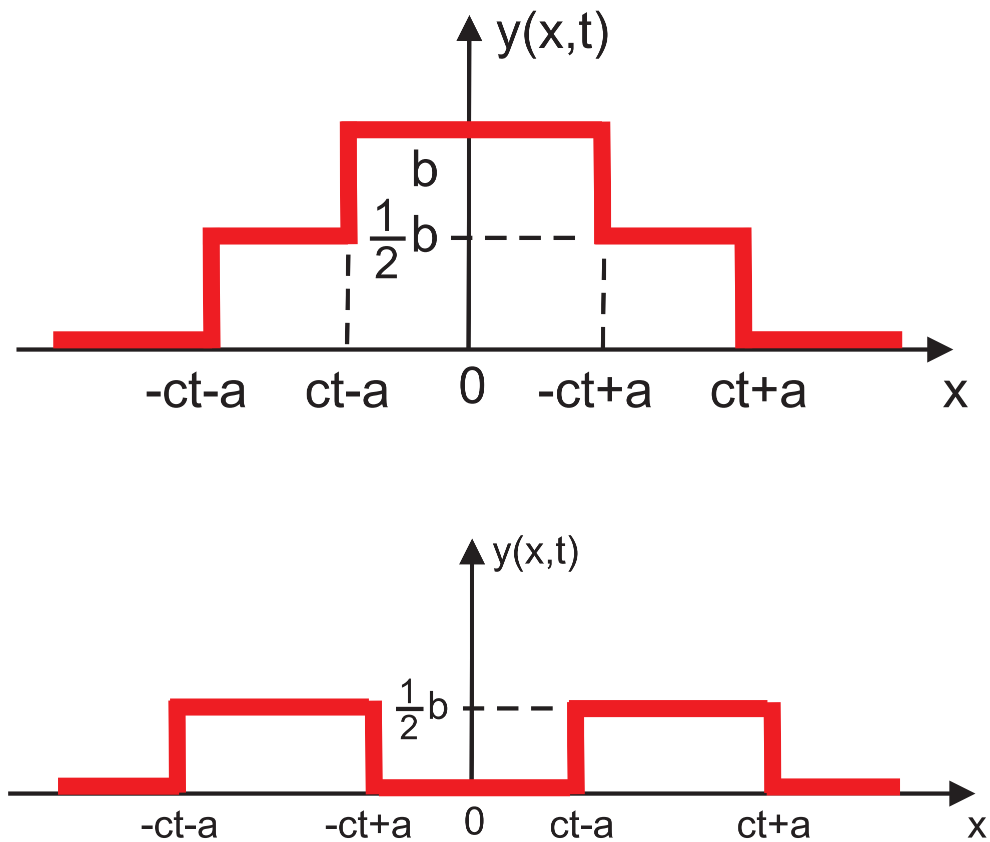

Figure 8: (a) $t < a/c$; (b) $t > a/c$

• This example illustrates well what Eq. (11) represents. The term $f(x-ct)$ has the same shape and size $\forall\, t$ (wave of permanent form); as t increases the profile moves to the right with speed c. Likewise $g(x+ct)$ is a profile of constant shape and size that moves to the left with speed c. Each is a travelling wave (or progressive wave). In the above example, the initial profile splits into two; one half travels to the right, one half to the left.

Example 2

Consider Eq. (12) with $\Phi(x) = a\sin(kx)$, where a and k are constants.

From Eq. (13) $\Rightarrow$

$$y(x,t) = \frac{1}{2}a\left\{\sin[k(x-ct)] + \sin[k(x+ct)]\right\}. \tag{17}$$

We shall revisit Eq. (17) soon.

• More general than Eq. (12) is the case when the string is also moving at $t = 0$.

$$y(x,0) = \Phi(x), \qquad y_t(x,0) = \Psi(x). \tag{18}$$

From Eqs. (11) and (18) we now have to choose $f(x)$ and $g(x)$ so that

$$f(x) + g(x) = \Phi(x), \qquad -cf'(x) + cg'(x) = \Psi(x).$$

The second gives

$$f'(x) - g'(x) = (-1/c)\Psi(x)$$

$\Rightarrow$

$$f(x) - g(x) = (-1/c)\int_d^x \Psi(s)ds,$$

where d is a constant. Thus

$$\begin{aligned} f(x) &= \frac{1}{2}\Phi(x) - \frac{1}{2c}\int_d^x \Psi(s)ds, \\ g(x) &= \frac{1}{2}\Phi(x) + \frac{1}{2c}\int_d^x \Psi(s)ds, \end{aligned}$$

and from Eq. (11) $\Rightarrow$

$$\begin{aligned} y(x,t) &= \frac{1}{2}\{\Phi(x-ct) + \Phi(x+ct)\} \\ &+ \frac{1}{2c}\int_d^{x+ct} \Psi(s)ds - \frac{1}{2c}\int_d^{x-ct} \Psi(s)ds \end{aligned}$$

$\Rightarrow$

$$y(x,t) = \frac{1}{2}\{\Phi(x-ct) + \Phi(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)ds. \tag{19}$$

Example 3

Given that $\Phi(x) = a\cos(kx)$, $\Psi(x) = -kca\sin(kx)$ in Eq. (18), find $y(x,t)$.

Solution

From Eq. (19),

$$\begin{aligned} y(x,t) &= \frac{a}{2}\{\cos(k(x-ct)) + \cos(k(x+ct))\} \\ &- \frac{ka}{2}\int_{x-ct}^{x+ct} \sin(ks)ds \\ &= \frac{a}{2}\{\cos(k(x-ct)) + \cos(k(x+ct))\} \\ &+ \frac{a}{2}[\cos(ks)]_{x-ct}^{x+ct} \\ &= a\cos(k(x+ct)) \end{aligned}$$

Thus the two terms in Eq. (19) combine so that the wave is purely travelling to the left.

Exercises for students:

[1] Show that Eq. (19) gives a wave travelling only to the left (i.e. $y = g(x+ct)$) if and only if $\Psi(x) = c\Phi'(x)$.
[2] What initial conditions give $y(x,t) = a\tanh(k(x-ct))$ for $-\infty < x < \infty$ and $\forall\ t \geq 0$?

1.4 Strings of finite length

- Now Eq. (17) can be written

$$(\text{since} \quad \sin A + \sin B = 2\sin\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right])$$

$$y(x,t) = a\sin(kx)\cos(kct) \tag{20}$$

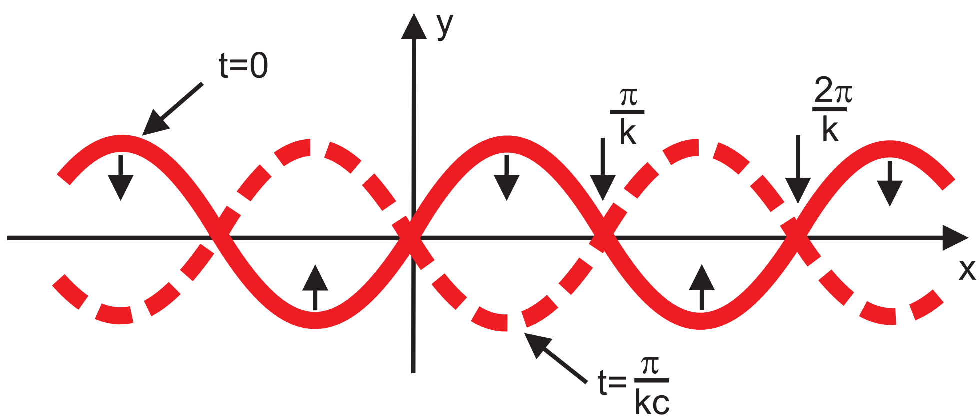

Figure 9: Standing wave

Thus y is always zero at $x = n\pi/k$.

Between $x = r_1\pi/k$ and $x = r_2\pi/k$ the string oscillates periodically in time.
Eq. (20) is an example of a standing wave, with a being the amplitude, k the wavenumber ($k > 0$), $2\pi/k$ the wavelength. The period of oscillation is $2\pi/kc$.

• Standing waves occur with a string of finite length L. Suppose the string is fixed at $x = 0$, $x = L$ (e.g., a piano wire or violin) so the solution of Eq. (5), the wave equation, must satisfy

$$y(0, t) = y(L, t) = 0. \tag{21}$$

We look for solutions of Eq. (5) of the form (separable solutions)

$$y(x, t) = X(x)T(t) \tag{22}$$

Substituting in Eq. (5) ⇒

$$c^2 X''T = X\ddot{T}$$

⇒

$$\frac{X''}{X} = \frac{1}{c^2}\left(\frac{\ddot{T}}{T}\right).$$

The LHS depends only on x, the RHS depends only on t so the equation can be true for $\forall\ (x, t)$ only if each side is a constant. There are three cases to consider.

[1] Constant $> 0 = k^2$

$\Rightarrow$

$$X'' = k^2 X$$

$\Rightarrow$

$$X = A\cosh(kx) + B\sinh(kx).$$

From Eq. (21) $\Rightarrow A = B = 0$. Not useful.

[2] Constant=0

$\Rightarrow$

$$X'' = 0$$

$\Rightarrow$

$$X = Ax + B.$$

From Eq. (21) $\Rightarrow A = B = 0$. Not useful.

[3] Constant $< 0 = -k^2$

$\Rightarrow$

$$X'' = -k^2 X, \quad \ddot{T} = -k^2 c^2 T. \tag{23}$$

First of Eq. (23) $\Rightarrow X = A\cos(kx) + B\sin(kx)$.

From Eq. (21):

$$y(0,t) = 0 \Rightarrow A = 0 \Rightarrow X = B\sin(kx)$$
$$y(L,t) = 0 \Rightarrow B\sin(kL) = 0.$$

For useful/interesting results we cannot have $B = 0$ which implies $sin(kl) = 0 \Rightarrow kL = n\pi$ (n$= 1, 2...$)

$\Rightarrow$

$$X = B_n sin(n\pi x/L)$$

and

$$\ddot{T} = -(n\pi c/L)^2 T.$$

$\Rightarrow$

$$T = \alpha\cos(n\pi ct/L) + \beta\sin(n\pi ct/L).$$

Thus a solution of Eq. (5) (wave equation) of the form Eq. (22) (separable solutions) satisfying Eq. (21) (fixed boundary) is

$$y = \sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \quad (n = 1, 2, 3...). \tag{24}$$

For each n, the solution in Eq. (24) is a periodic wave [like Eq. (20)] with period $2\pi L/n\pi c = 2L/nc$.

We often rewrite

$$\begin{matrix} \cos(n\pi ct/L) & & cos(\omega_n t) \\ & \text{as} & \\ \sin(n\pi ct/L) & & sin(\omega_n t) \end{matrix}$$

where ω_n is the angular frequency:

$$\omega_n = \frac{n\pi c}{L}. \tag{25}$$

Each of the solutions in Eq. (24) is a normal mode of vibration.

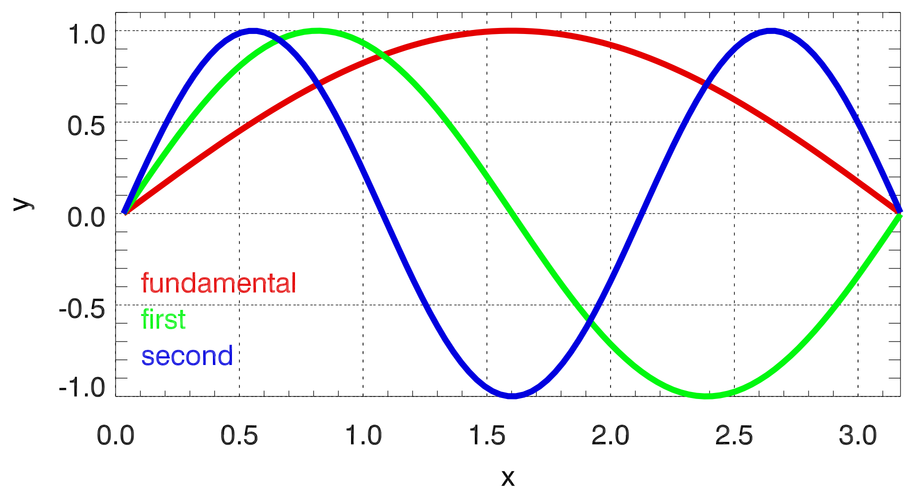

Figure 10: Standing fundamental, 1st, and 2nd harmonics

• Now Eq. (5) is a linear equation so any linear combination of the solutions in Eq. (24) is also a solution. This is the principle of superposition. Thus

$$y = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \tag{26}$$

is a solution of Eq. (5) satisfying Eq. (21). It is in fact the general solution of Eq. (5)-(21); the constants α_n, β_n are determined by the initial conditions (see Chapter Two).

Question: In general, is this solution periodic in time? Explain your answer.

1.5 Some technical remarks

- Consider the real part, $\Re$, of the complex quantity

$$A\exp[i(kx-\omega t)],$$

where k and ω are real but

$$A = A_r + iA_i$$

is complex. Now

$$\begin{aligned}\Re\{A\exp[i(kx-\omega t)]\} &= A_r\cos(kx-\omega t) - A_i\sin(kx-\omega t)\\ &= \sqrt{A_r^2+A_i^2}\cos[(kx-\omega t)+\epsilon]\end{aligned}$$

where

$$\cos\epsilon = A_r/\sqrt{A_r^2+A_i^2}, \qquad \sin\epsilon = A_i/\sqrt{A_r^2+A_i^2}.$$

We shall consider situations in which the dependent variable, say ϕ, has the form

$$\phi = \alpha\cos[(kx-\omega t)+\epsilon]$$

(or with sin instead of cos).
Note: $\phi = \sin kx[(-\alpha\sin\epsilon)\cos\omega t + (\alpha\cos\epsilon)\sin\omega t] - \cos kx[(-\alpha\cos\epsilon)\cos\omega t + (\alpha\sin\epsilon)\sin\omega t]$, and the first term is equivalent to Eq. (24).

In linear problems it is often convenient to write (A complex; k, ω real)

$$\phi = A \exp[i(kx - \omega t)]; \tag{27}$$

we do of course really mean the real part of Eq. (27) but many problems can be solved most easily by working directly with Eq. (27) and only taking the real part right at the end.

In Eq. (27), k is again the wavenumber and ω is the angular frequency.

To satisfy the 1D wave equation Eq. (5), $\omega = kc$. The period is $2\pi/\omega$ and the frequency is $\omega/2\pi$. The frequency, measured in s^{-1} (Hz, hertz), is the number of complete oscillations that the wave makes during 1 sec at a fixed position. Finally,

$$|A| = \sqrt{A_r^2 + A_i^2}$$

is the amplitude. Eq. (27) is a periodic or harmonic wave.