2 Use of Fourier Series

2.1 Aim of Chapter

• In § 1.4, we saw that a solution of the PDE for $y(x,t)$

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad \text{with} \qquad y(0,t) = y(L,t) = 0, \qquad (1)$$

is Eq (1.24), viz.

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right]. \qquad (2)$$

Eq. (2) is in fact the general solution of (1). Our aim is to show how the constants $\{\alpha_n\}$ and $\{\beta_n\}$ can be determined, and to indicate some extensions.

• The constants $\{\alpha_n\}$ and $\{\beta_n\}$ are determined uniquely by the initial conditions, i.e. the value of $y(x,0)$ and $\dot{y}(x,0)$ (or, more generally, by the values of $y(x,t_0)$, $\dot{y}(x,t_0)$ for any t_0). In any particular motion, the values of $y(x,0)$ and $\dot{y}(x,0)$ can be chosen independently of one another.

- We note from Eq. (2) that

$$\dot{y}(x,t) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \times \left[-\alpha_n \sin\left(\frac{n\pi ct}{L}\right) + \beta_n \cos\left(\frac{n\pi ct}{L}\right)\right]. \quad (3)$$

- Thus, from Eqs. (2) and (3)

$$y(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right), \quad (4a)$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \beta_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \quad (4b)$$

2.2 Worked Examples

Example 1

Find $\{\alpha_n\}$, $\{\beta_n\}$ for the case of a plucked string of length L, with its ends fixed, released from rest when the mid-point is drawn aside through a distance h. Thus

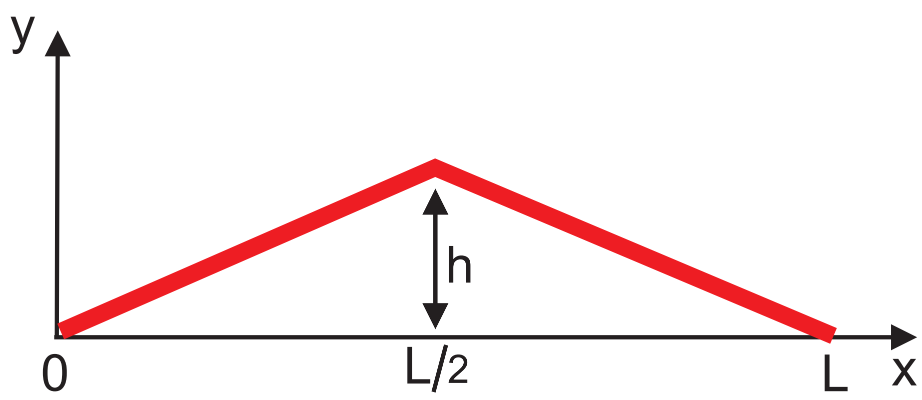

Figure 1: Plucked string with length L

$$y(x,0) = \begin{cases} \frac{2h}{L}x & (0 \le x \le \frac{1}{2}L) \\ \frac{2h}{L}(L-x) & (\frac{1}{2}L \le x \le L) \end{cases} \tag{5}$$

and

$$\dot{y}(x,0) = 0. \tag{6}$$

Solution

- Comparing Eqs. (6) and (4b), we can reconcile them by taking

$$\beta_n = 0. \tag{7}$$

It remains to reconcile Eqs. (5) and (4a). The key is to multiply Eq. (4a) by

$$\sin\left(\frac{m\pi x}{L}\right)$$

and integrate from $x = 0$ to $x = L$. Thus

$$\int_0^L y(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\alpha_n\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{8}$$

- Consider I_{mn}, where

$$I_{mn} = \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{9}$$

Question: How to proceed with the evaluation of I_{mn}?

Now

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

for $\forall$ A and B, so the integrand in Eq. (9) is

$$\frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right].$$

Since m and n are positive integers, there are two possible cases.

$m \neq n$:

$$I_{mn} = \frac{l}{2\pi}\left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{(m-n)} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{(m+n)}\right]_0^L = 0$$

$m = n$:

$$I_{mn} = \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2m\pi x}{L}\right)\right] dx = \frac{L}{2}.$$

$\Rightarrow$

$$I_{mn} = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n. \end{cases} \qquad (10)$$

• Hence the RHS of Eq. (8) reduced to $\frac{L}{2}\alpha_m$, and Eq. (8) can be then rewritten

$$\alpha_m = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{m\pi x}{L}\right)dx, \tag{11}$$

where Eq. (11) is a general formula. In our particular case, use of Eq. (5) gives

$$\begin{aligned}
\alpha_m &= \frac{4h}{L^2}\int_0^{L/2} x\sin\left(\frac{m\pi x}{L}\right)dx \\
&+\frac{4h}{L^2}\int_{L/2}^{L}(L-x)\sin\left(\frac{m\pi x}{L}\right)dx, \\
&= \left\{\frac{4h}{Lm\pi}\left[-x\cos\left(\frac{m\pi x}{L}\right)\right]_0^{L/2}\right. \\
&\left.+\frac{4h}{Lm\pi}\int_0^{L/2}\cos\left(\frac{m\pi x}{L}\right)dx\right\}+ \\
&+\left\{\frac{4h}{Lm\pi}\left[-(L-x)\cos\left(\frac{m\pi x}{L}\right)\right]_{L/2}^{L}\right. \\
&\left.-\frac{4h}{Lm\pi}\int_{L/2}^{L}\cos\left(\frac{m\pi x}{L}\right)dx\right\}, \\
&= -\frac{2h}{m\pi}\cos\left(\frac{m\pi}{2}\right)+\frac{2h}{m\pi}\cos\left(\frac{m\pi}{2}\right)+\frac{4h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right) \\
&+\frac{4h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)=\frac{8h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right).
\end{aligned}$$

Thus

$$\alpha_m = \begin{cases} 0 & (m = 2p) \\ \frac{8h(-1)^p}{\pi^2(2p+1)^2} & (m = 2p+1). \end{cases} \quad (12)$$

• Use of Eqs. (7) and (12) upon substitution into Eq. (2) gives

$$y(x,t) = \frac{8h}{\pi^2} \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)^2} \sin\left(\frac{(2p+1)\pi x}{L}\right) \times \cos\left(\frac{(2p+1)\pi ct}{L}\right). \quad (13)$$

Example 2

Find $\{\alpha_n\}$, $\{\beta_n\}$ for the case of a string of length L, given that Eq. (1) holds and in addition $y(x,0) = 0$ and $\dot{y}(x,0) = 4Vx(L-x)/L^2$.

Solution

In this case $y(x,0) = 0 \Rightarrow$

$$\alpha_n = 0. \tag{14}$$

Then from Eq. (4b) $\Rightarrow$

$$\int_0^L \frac{4Vx(L-x)}{L^2} \sin\left(\frac{m\pi x}{L}\right) dx = \beta_m \left(\frac{m\pi c}{L}\right) \frac{L}{2},$$

using the same technique that leads from Eq. (8) to Eq. (11).

Hence

$$
\begin{aligned}
\beta_m \frac{m\pi c}{2} &= \frac{4V}{L^2}\left\{\underbrace{\left[-\frac{L}{m\pi}x(L-x)\cos\left(\frac{m\pi x}{L}\right)\right]_0^L}_{=0}\right. \\
&\quad \left. + \frac{L}{m\pi}\int_0^L (L-2x)\cos\left(\frac{m\pi x}{L}\right)dx\right\}, \\
&= \frac{4V}{L^2}\left\{0 + \frac{L}{m\pi}\int_0^L (L-2x)\cos\left(\frac{m\pi x}{L}\right)dx\right\}.
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\beta_m &= \frac{8V}{m^2\pi^2 cL}\left\{\left[\frac{L}{m\pi}(L-2x)\sin\left(\frac{m\pi x}{L}\right)\right]_0^L\right. \\
&\quad \left. +\frac{2L}{m\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right)dx\right\}, \\
&= \frac{16VL}{m^4\pi^4 c}\left[\cos\left(\frac{m\pi x}{L}\right)\right]_L^0 \\
&= \frac{16VL}{m^4\pi^4 c}[1-(-1)^m].
\end{aligned}
$$

Thus

$$\beta_m = \begin{cases} 0 & (m = 2p) \\ \frac{32VL}{\pi^4 c(2p+1)^4} & (m = 2p+1) \end{cases}, \tag{15}$$

and so

$$y(x,t) = \frac{32VL}{\pi^4 c} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} \sin\left(\frac{(2p+1)\pi x}{L}\right) \times \sin\left(\frac{(2p+1)\pi ct}{L}\right). \tag{16}$$

Note: The sketches on the hand-out show how series like Eqs. (13) and (16) converge. Discontinuities, like that in the gradient of $y(x,0)$ at $x = L/2$ in Example 1, cause the coefficients in the series to decrease less rapidly with n when then there are no discontinuities. Compare the rates of fall-off with p of the coefficients in Eqs. (13) and (16). Thus Eq. (16) indicates a "purer" tone than Eq. (13).

2.3 Energy

• Consider a string occupying $0 \le x \le L$ with $y(0,t) = 0$, $y(L,t) = 0$, and consider the normal mode Eq. (1.24)

$$y = \sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\}.$$

We rewrite this in the form (with $A_n \ge 0$, $0 \le \epsilon_n \le 2\pi$):

$$y = A_n \cos\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \sin\left(\frac{n\pi x}{L}\right). \qquad (17)$$

In Eq. (17) A_n is the amplitude and ϵ_n is the phase.

Note:

$$\begin{aligned} A_n \cos\{n\pi ct/L + \epsilon_n\} &= A_n \cos\epsilon_n \cos(n\pi ct/L) \\ &- A_n \sin\epsilon_n \sin(n\pi ct/L) \end{aligned}$$

so Eqs. (1.24) and (17) are the same provided

$$A_n \cos\epsilon_n = \alpha_n, \qquad A_n \sin\epsilon_n = -\beta_n.$$

$\Rightarrow$

$$A_n^2(\cos^2\epsilon_n + \sin^2\epsilon_n) = \alpha_n^2 + \beta_n^2$$

$\Rightarrow$

$$A_n = +\sqrt{\alpha_n^2 + \beta_n^2}, \qquad \tan\epsilon_n = -\beta_n/\alpha_n.$$

• By Eq. (1.7), the kinetic energy T_n associated with Eq. (17) is

$$T_n = \frac{1}{2}\rho A_n^2 \left(\frac{n\pi c}{L}\right)^2 \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx,$$

$\Rightarrow$

$$T_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \qquad (18)$$

Likewise, by Eq. (1.8), the potential energy V_n associated with Eq. (17) is

$$\begin{aligned} V_n &= \frac{1}{2} F A_n^2 \left(\frac{n\pi}{L}\right)^2 \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx, \\ &= \frac{F\pi^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \end{aligned}$$

From Eq. (1.6) $F = \rho c^2$, $\Rightarrow$

$$V_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \qquad (19)$$

The total energy therefore $E_n = T_n + V_n$ is given by

$$E_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} = \frac{\rho L}{4}\omega_n^2 A_n^2, \qquad \omega_n = \frac{n\pi c}{L}, \tag{20}$$

where ω_n is the angular frequency of this normal mode.

$\Rightarrow$ $E_n \propto A_n^2$ and $E_n \propto \omega_n^2$.

$E_n \propto A_n^2$ indicates that a much bigger proportion of the total energy is contained in the first few modes of, say, Eq. (16) than in the same number of modes of, say, Eq. (13). **Check it!**

• Now consider the general motion given by Eq. (1.24). Because, for $m \neq n$,

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0,$$

(the first leads to Eq. (10b) and the second is proved likewise), it follows immediately that

$$T = \sum_{n=1}^{\infty} T_n = \frac{\rho\pi^2c^2}{4L}\sum_{n=1}^{\infty} n^2A_n^2\sin^2\left\{\frac{n\pi ct}{L}+\epsilon_n\right\},$$

$$V = \sum_{n=1}^{\infty} V_n = \frac{\rho\pi^2c^2}{4L}\sum_{n=1}^{\infty} n^2A_n^2\cos^2\left\{\frac{n\pi ct}{L}+\epsilon_n\right\},$$

$$E = T + V = \frac{\rho\pi^2c^2}{4L}\sum_{n=1}^{\infty} n^2A_n^2. \tag{21}$$

• Apply to **Example 1** in § (2.2). From Eq. (12) we have

$$\begin{aligned} A_{2n} &= 0 \\ A_{2n+1} &= \frac{8h}{\pi^2(2n+1)^2}. \end{aligned}$$

Thus, from the last of Eq. (21),

$$E = \frac{\rho\pi^2c^2}{4L}\sum_{n=0}^{\infty}\frac{64(2n+1)^2h^2}{\pi^4(2n+1)^4},$$

and thus ⇒

$$E = \frac{16\rho h^2c^2}{\pi^2L}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \tag{22}$$

Amazingly it turns out that we can evaluate the infinite series in Eq. (22) by using Eq. (13)! We are **given** that $y(L/2,0) = h$ (see Eq. 5). Thus, putting $x = L/2$ and $t = 0$ in Eq. (13), we find

$$\begin{aligned} h &= \frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left[\frac{(2n+1)\pi}{2}\right] \\ &= \frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\times(-1)^n}{(2n+1)^2}. \end{aligned}$$

Thus

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}, \qquad (23)$$

and Eq. (22) becomes

$$E=\frac{2\rho h^2c^2}{L}. \qquad (24)$$

We can check this result from the initial conditions when $T=0$ and

$$\begin{aligned} V &= \frac{1}{2}F\left[\int_0^{L/2}\left(\frac{2h}{L}\right)^2 dx+\int_{L/2}^{L}\left(\frac{-2h}{L}\right)^2 dx\right]=\frac{2Fh^2}{L} \\ &= \frac{2\rho h^2c^2}{L}, \end{aligned}$$

using Eq. (1.8). Thus

$$E|_{t=0}=0+\frac{2\rho h^2c^2}{L}.$$

• As a matter of fact, this worked example corresponds quite closely to a violin string plucked at its mid-point.

The fundamental frequency, or pitch, is

$$\frac{\pi c}{L}\frac{1}{2\pi} = \frac{c}{2L},$$

but overtones with frequencies

$$\frac{3c}{2L}, \quad \frac{5c}{2L}, \quad \cdots$$

are generated. The note heard by a listener depends on the amplitudes of the overtones; the note is not pure but the (relatively) rapid fall-off of the amplitudes means that the note is purer than that of many musical instruments, particularly the piano.

If the string had been bowed at some other point than its center, the amplitude of the overtones would have been different and thus tone would have been changed.

2.4 Two (different) extensions

2.4.1 Fourier transforms

Series like Eqs. (4a) and (4b), and (perhaps!) Eqs. (13) and (16) are known as **Fourier Series** after the great French scientist and mathematician (Jean Baptiste) Joseph Fourier (1768-1830).

The methods used in this chapter are capable of extension in many different directions. The only one I want to draw attention to here is the following. We have seen Eq. (1.17) that

$$\Phi = A \exp[ik(x - ct)]$$

is a solution of the 1D wave equation for any value of the constant k. So therefore is

$$A(k) \exp[ik(x - ct)]$$

for any function $A(k)$ and, also,

$$\int_{-\infty}^{\infty} A(k) \exp[ik(x - ct)]dk.$$

This leads/is related to Fourier Transforms or Fourier Analysis.[1]

[1] You might enjoy looking at "Fourier Analysis" by T.W.Kövner, CUP (1988).

2.4.2 2D wave equation

Consider a membrane, e.g. the surface of a drum. Let (x, y) denote position in the membrane and $z = z(x, y, t)$ be its transverse displacement. It can be easily shown that the governing equation for z is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right). \tag{25}$$

We can find separable solutions - see § (1.4)- of the form

$$z = X(x)Y(y)T(t).$$

But for a drum it is more natural to use polar coordinates (r, θ) with

$$x = r\cos\theta, \quad y = r\sin\theta$$

when Eq. (25) becomes (for details see notes at the end)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right] \tag{26}$$

- As in § (1.4) we seek separable solutions of the form

$$z = R(r)\Theta(\theta)T(t),$$

but we shorten the process by looking for normal modes with

$$T \propto e^{i\omega t}.$$

$\Rightarrow$

$$\Theta\left[r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + R\frac{d^2\Theta}{d\theta^2} = -\frac{\omega^2}{c^2}r^2 R\Theta$$

$\Rightarrow$

$$\frac{1}{R}\left[r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + k^2r^2 = 0, \qquad (27)$$

where

$$k = \frac{\omega}{c} \qquad (28)$$

- Suppose

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \text{const} := -n^2$$

$\Rightarrow$

$$\Theta \propto e^{in\theta}.$$

In practice, we must have

$$\Theta(\theta) = \Theta(\theta + 2\pi) \Rightarrow n \in N^+$$

The equation for R becomes

$$r^2 R'' + rR' + (k^2 r^2 - n^2)R = 0,$$

and the change of variable $\xi = kr$ gives

$$\xi^2 \frac{\mathbf{d^2R}}{\mathbf{d}\xi^2} + \xi \frac{\mathbf{dR}}{\mathbf{d}\xi} + (\xi^2 - \mathbf{n^2})\mathbf{R} = \mathbf{0}. \tag{29}$$

This is known as **Bessel's equation of order** n.

We now consider only the case $n = 0 \Rightarrow$ no θ variation! The only solution of Eq. (29) that is bounded at $r = 0$ is

$$R \propto J_0(\xi) = J_0(kr), \tag{30}$$

(see hand-out).

Assume, as with a drum, that the membrane is fixed at

$$r = a \Rightarrow R = 0 \text{ when } r = a$$

$\Rightarrow$

$$J_0(ka) = 0 \Rightarrow k = \frac{\lambda_m}{a}$$

where λ_m is the m-th root of $J_0(\xi)$.

- Thus

$$z = A_m J_0\left(\frac{\lambda_m r}{a}\right) e^{i\lambda_m ct/a}$$

and the **general solution**, independent of θ, is

$$\mathbf{z} = \sum_{\mathbf{n}=1}^{\infty} \mathbf{A_m J_0}\left(\frac{\lambda_\mathbf{m} \mathbf{r}}{\mathbf{a}}\right) \mathbf{e}^{\mathbf{i}\lambda_\mathbf{m}\mathbf{ct/a}}. \tag{31}$$

Exercise

Find $\{A_m\}$ by similar methods to those in § (2.2). (Note, there is an orthogonality relationship.)

Notes on 2D cylindrical wave equation

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\begin{aligned}\therefore\quad z_r &= \cos\theta z_x + \sin\theta z_y \\ z_\theta &= -r\sin\theta z_x + r\cos\theta z_y\end{aligned}$$

$$\begin{aligned}\therefore\quad z_x &= \cos\theta z_r - \frac{\sin\theta}{r} z_\theta \\ z_y &= \sin\theta z_r + \frac{\cos\theta}{r} z_\theta\end{aligned}$$

$$\begin{aligned}\therefore\quad z_{xx} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta z_r - \frac{\sin\theta}{r} z_\theta\right) \\ &= \cos^2\theta z_{rr} + \frac{\cos\theta\sin\theta}{r^2} z_\theta - \frac{\cos\theta\sin\theta}{r} z_{r\theta} \\ &\quad + \frac{\sin^2\theta}{r} z_r - \frac{\sin\theta\cos\theta}{r} z_{r\theta} \\ &\quad + \frac{\sin\theta\cos\theta}{r^2} z_\theta + \frac{\sin^2\theta}{r^2} z_{\theta\theta} \\ &= \cos^2\theta z_{rr} + \frac{\sin^2\theta}{r} z_r + \frac{\sin^2\theta}{r^2} z_{\theta\theta} \\ &\quad - \frac{2\cos\theta\sin\theta}{r^2} z_{r\theta} + \frac{2\cos\theta\sin\theta}{r^2} z_\theta.\end{aligned}$$

Likewise, after algebra:

$$\begin{aligned} z_{yy} &= \sin^2\theta z_{rr} + \frac{\cos^2\theta}{r} z_r + \frac{\cos^2\theta}{r^2} z_{\theta\theta} \\ &\quad + \frac{2\cos\theta\sin\theta}{r^2} z_{r\theta} - \frac{2\cos\theta\sin\theta}{r^2} z_\theta. \end{aligned} \tag{32}$$

Therefore

$$\begin{aligned} z_{xx} + z_{yy} &= z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta}, \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r z_r) + \frac{1}{r^2} z_{\theta\theta}. \end{aligned}$$

Convergency of Fourier series

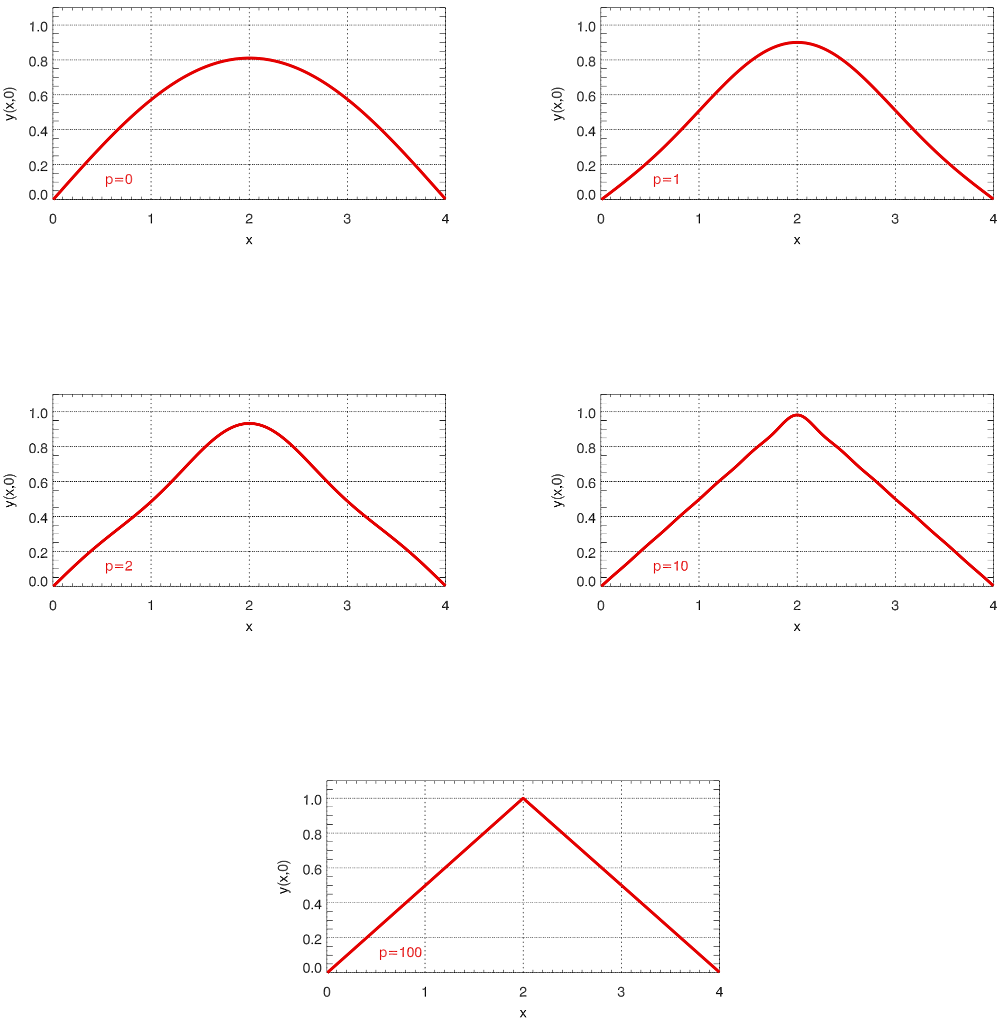

Figure 1: Convergency of Eq. (2.13) for p=0, 1, 2, 10 and 100 at t=0 (Example 1)

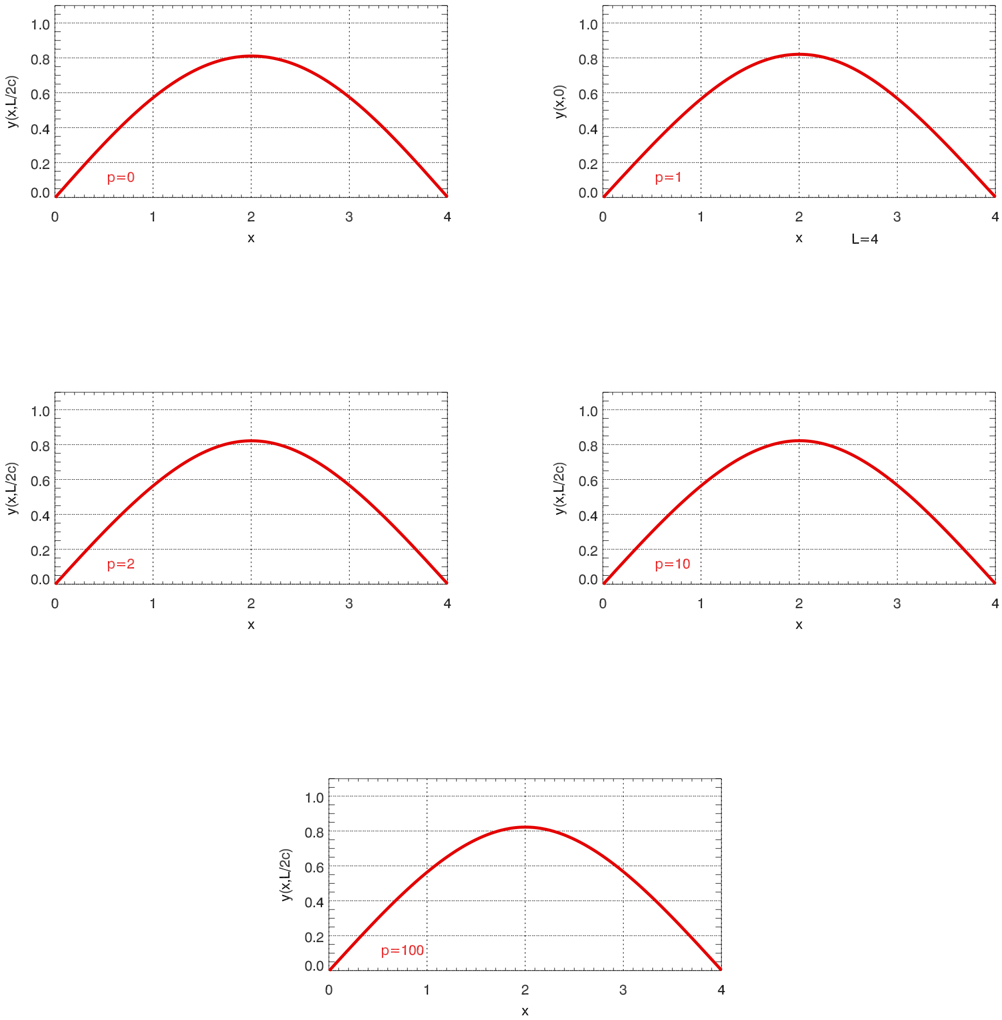

Figure 2: Convergency of Eq. (2.16) for p=0, 1, 2, 10 and 100 at $t = 0$ (Example 2)

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BESSEL FUNCTIONS

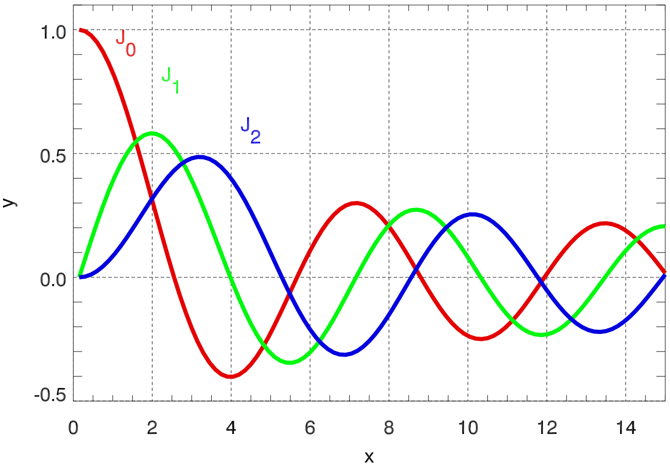

Figure 1: Bessel function J_n

- $y = J_n(x) \quad (n = 0, 1, 2, ..)$ is a solution of **Bessel's equation** of order n. This is (*see Eq. (2.29) in Notes*):

$$x^2 y'' + xy' + (x^2 - n^2)\, y = 0.$$

- $J_n(x)$ is the **Bessel function of order n** defined precisely by the infinite series

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p x^{n+2p}}{2^{n+2p}\, p!\, (n+p)!}.$$

  This can be shown to satisfy Bessel's equation.

- The general solution of Bessel's equation of order n is unbounded as $x \to 0$. The most general solution that is bounded as $x \to 0$ is $y = AJ_n(x)$ where A is an arbitrary constant.

- As the sketches illustrate, $J_n(x)$ has an infinite number of zeros.

- Let α_m be the mth zero of $J_0(x)$. To good approximation

$$\alpha_1 = 2.405, \quad \alpha_2 = 5.520, \quad \alpha_3 = 8.654$$

$$\alpha_m \approx \left(m - \frac{1}{4}\right) \pi \quad \text{for large } m$$

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