3 Sound Waves and the Equation of Continuity

3.1 One-dimensional sound waves

• In general most phenomena associated with sound propagation can be described well by assuming that air (or the underlying fluid) has uniform density ρ_0 and uniform pressure p_0 in the undisturbed state, and that sound generation causes small changes in density ρ and pressure p, and a small velocity $\mathbf{u}$.

• We shall suppose that p and ρ are uniquely related:

$$p = p(\rho) \leftrightarrow \rho = \rho(p) \tag{1}$$

• We begin by assuming that the motion is 1D

$\Rightarrow$

$$\mathbf{u} = u(x,t)\mathbf{i}, \quad p = p(x,t), \quad \rho = \rho(x,t)$$

• Now ρ and u are not independent since mass must be conserved.

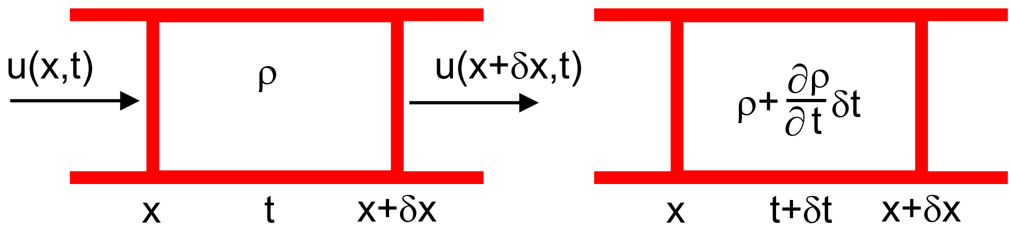

Figure 1: Conservation of mass

Consider a small tube of length δx and cross-sectional area A.

At time t the mass of fluid in the tube is

$$\rho(x,t)\delta x A$$

and at time $t+\delta t$ it is

$$[\rho(x,t)+\delta\rho]\delta x A = \rho(x,t)\delta x A + \frac{\partial \rho(x,t)}{\partial t}\delta x \delta t A.$$

The mass has increased by an amount

$$\frac{\partial \rho}{\partial t}\delta x \delta t A.$$

This is due to mass flowing into the tube: in time δt this is equal to

$$\left\{\rho u|_{(x,t)} - \rho u|_{(x+\delta x,t)}\right\}\delta t A = -\frac{\partial}{\partial x}(\rho u)\delta x \delta t A$$

$$= -\rho\frac{\partial u}{\partial x}\delta x \delta t A - u\frac{\partial \rho}{\partial x}\delta x \delta t A.$$

But u and $\partial\rho/\partial x$ are small (we linearize) $\Rightarrow$ this is equal to

$$= -\rho_0\frac{\partial u}{\partial x}\delta x \delta t A$$

to highest order. Thus

$$\frac{\partial \rho}{\partial t} = -\rho_0\frac{\partial u}{\partial x} \tag{2}$$

• Now we apply N2 to the fluid in the small tube. From Fig. (2) the force on the fluid is

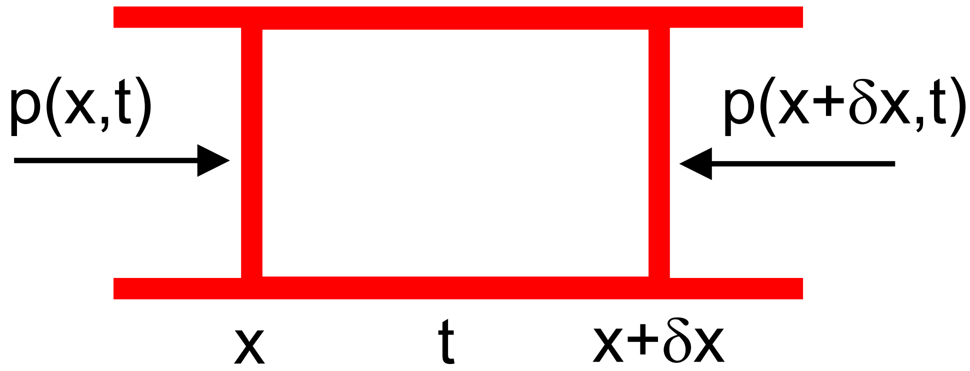

Figure 2: Forces acting on a fluid element with length δx

$$[p(x,t)A - p(x+\delta x,t)A] = -\frac{\partial p}{\partial x}\delta x A,$$

neglecting gravity and this is equal to

$$\rho \delta x A \frac{\partial u}{\partial t} \approx \rho_0 \frac{\partial u}{\partial t} \delta x A$$

by N2. Thus

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}. \qquad (3)$$

- Now, by Eq. (1),

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \\ &= \left(\frac{dp}{d\rho}\right)_{\rho=\rho_0}\frac{\partial \rho}{\partial x} \\ &:= c^2\frac{\partial \rho}{\partial x},\end{aligned}$$

where

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right). \tag{4}$$

Thus, from Eq. (2),

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0\frac{\partial^2 u}{\partial x \partial t} = -\rho_0\frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}c^2\frac{\partial \rho}{\partial x}\right),$$

i.e.

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\frac{\partial^2 \rho}{\partial x^2}.$$

Likewise, the same equation holds for u (and p). Thus

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \& \quad \frac{\partial^2 \rho}{\partial t^2} = c^2\frac{\partial^2 \rho}{\partial x^2} \tag{5}$$

• For sound in gases, the appropriate relationship Eq. (1) between p and ρ is the adiabatic law*

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \Rightarrow \frac{\partial p}{\partial \rho} = \frac{\gamma p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma - 1}.$$

(*NB No heat exchange ⇒ change occurs quickly. Newton derived Eq. (5), but assumed that p and ρ were related by the isothermal law

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}$$

Boyle's Law. ⇒

$$c = \left(\frac{p_0}{\rho_0}\right)^{1/2} \approx 280\,\mathrm{m\ s^{-1}}$$

Correct expression due to Laplace.)

By Eq. (4), the velocity of sound in air is

$$c = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} \approx 330\,\mathrm{m\ s^{-1}} \tag{6}$$

at standard temperature and pressure. (T=288 K; $\gamma = 1.4$, $p_0 \approx 1.013 \times 10^5\ \mathrm{Nm^{-2}}$, $\rho_0 \approx 1.293\ \mathrm{kgm^{-3}}$). Eq. (6) agrees well with experiments.

The same analysis applies to water where experiments show that

$$(p - p_0) = \kappa(\rho - \rho_0)/\rho_0$$

$\Rightarrow$

$$\frac{dp}{d\rho} = \frac{\kappa}{\rho_0}$$

$\Rightarrow$

$$c \approx 1.430 \times 10^3 \text{ m s}^{-1}$$

$(\kappa \approx 2.045 \times 10^9 \text{ kgm}^{-1}\text{s}^{-2},\ \rho_0 \approx 10^3 \text{ kg m}^{-3})$

- One dimensional sound waves can be treated by the same mathematical methods as (one-dimensional) waves on strings. Note however that in strings the motion is transverse; in sound waves it is longitudinal.

3.2 The Equation of continuity of a fluid

• In reality, sound propagates in 3D (although the 1D result can be applied approximately to e.g. pipes). We now assume the fluid velocity $\mathbf{u}$ is still small but with three non-zero components:

$$\mathbf{u} = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}. \tag{7}$$

We must again ensure that mass is conserved - the resulting equation, viz. Eq. (8), is known as the equation of continuity.

• Consider a small cuboid with sides parallel to the axes, and of lengths $\delta x, \delta y, \delta z$.

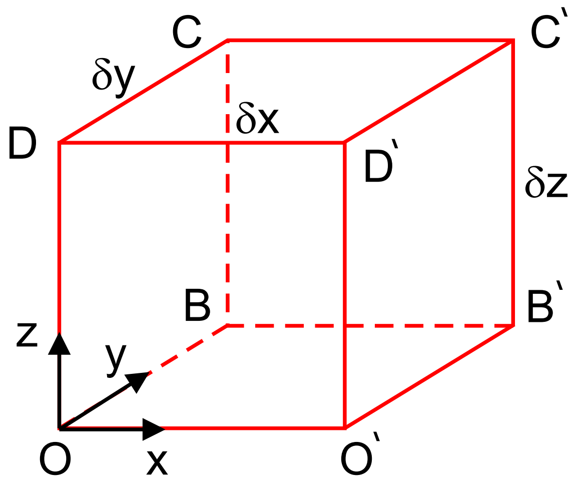

Figure 3: Three-dimensional fluid element $\delta x \delta y \delta z$

As above, the increase in the mass within this cuboid between times t and δt is

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z \delta t.$$

This is equal to the mass flowing into the cuboid, which is equal to

$$\begin{aligned} & \left\{\rho u|_{OBCD} - \rho u|_{O'B'C'D'}\right\} \delta y \delta z \delta t \\ + & \left\{\rho v|_{OO'D'D} - \rho v|_{BB'C'C}\right\} \delta z \delta x \delta t \\ + & \left\{\rho w|_{OO'B'B} - \rho w|_{DD'C'C}\right\} \delta x \delta y \delta t \\ \approx & -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \delta x \delta y \delta z \delta t. \end{aligned}$$

Thus

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{8}$$

Eq. (8) is exact since no assumption of smallness has yet been made. This is the equation of continuity for any fluid.

3.3 Three-dimensional sound waves

• When $\rho - \rho_0$ and u, v, w are small, Eq. (8) becomes

$$\frac{\partial \rho}{\partial t} + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \underbrace{(\rho - \rho_0) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_{\text{2nd order}}$$

$$\underbrace{+u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}}_{\text{2nd order}} = 0.$$

Thus

$$\frac{\partial \rho}{\partial t} = -\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \tag{9}$$

replaces Eq. (2) for sound waves.

• N2 gives three equations like Eq. (3)[1] $\Rightarrow$:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{10}$$

or, using Eq. (4):

$$\frac{\partial u}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial y}, \quad \frac{\partial w}{\partial t} = -\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial z} \tag{11}$$

[1] The LHS of Eq. (10) - and Eq. (3) earlier - involve an assumption of u, v, w being small. See § (4.1)

Thus, from Eq. (9)

$$\begin{aligned}\frac{\partial^2\rho}{\partial t^2} &= -\rho_0\left(\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 v}{\partial y\partial t}+\frac{\partial^2 w}{\partial z\partial t}\right)\\ &= -\rho_0\left\{\frac{\partial}{\partial x}\left(-\frac{c^2}{\rho_0}\frac{\partial\rho}{\partial x}\right)\right.\\ &\qquad\left.+\frac{\partial}{\partial y}\left(-\frac{c^2}{\rho_0}\frac{\partial\rho}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{c^2}{\rho_0}\frac{\partial\rho}{\partial z}\right)\right\}.\end{aligned}$$

Hence

$$\frac{\partial^2\rho}{\partial t^2} = c^2\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\frac{\partial^2\rho}{\partial z^2}\right). \tag{12}$$

This is the 3D wave equation with speed c.

- In most circumstances $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad w = \frac{\partial\phi}{\partial z}, \tag{13}$$

and ϕ is the velocity potential. It is shown in S3 Q3 that ϕ also satisfies the same eqn. (so therefore do u, v, w):

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right). \tag{14}$$