5 Modelling Traffic Flow

5.1 Quasi-linear first-order PDEs

• We consider only two independent variables (x, y) and an unknown $z = z(x, y)$ satisfying a first order PDE. This PDE is quasi-linear if it is linear in its highest order terms, i.e.

$$z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad z_y = \frac{\partial z}{\partial y}.$$

Thus

$$\begin{aligned} zz_x + z_y &= 0 \quad \text{is quasi-linear (and non-linear)} \\ (z_x)^2 + z_y &= 0 \quad \text{is not quasi-linear.} \end{aligned}$$

The most general first-order quasi-linear PDE is:

$$Pz_x + Qz_y = R \tag{1}$$

where

$$P = P(x, y, z),\ Q = Q(x, y, z),\ R = R(x, y, z) \tag{2}$$

are given continuous functions.

• Consider the family of curves in the (x, y) plane satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q}{P} \quad \text{or} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{P}{Q} \quad \text{or} \quad \frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}.$$

Suppose z is known at a point $A(x, y)$. There is one curve Γ_A of this family through A, and along Γ_A

$$\mathrm{d}z = z_x \mathrm{d}x + z_y \mathrm{d}y = \left(z_x + \frac{Q}{P} z_y\right) \mathrm{d}x = \frac{R}{P} \mathrm{d}x \tag{3}$$

using Eq. (1). Hence $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{R}{P}$ along Γ_A, and so:

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}. \tag{4}$$

Eqs. (4) are known as the associated equations for Eq. (1), and are equivalent to Eq. (1). For let each term in Eq. (4) be $\mathrm{d}s$, so $\mathrm{d}x = P\mathrm{d}s$, $\mathrm{d}y = Q\mathrm{d}s$, $\mathrm{d}z = R\mathrm{d}s$. Substitute in Eq. (3) to get

$$R\mathrm{d}s = P z_x \mathrm{d}s + Q z_y \mathrm{d}s \Rightarrow P z_x + Q z_y = R,$$

i.e. Eq. (1).

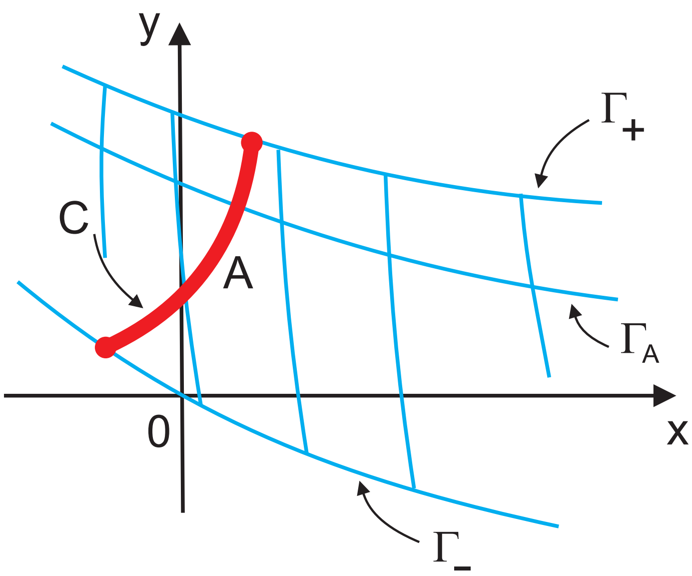

Figure 1: Characteristic curves

• Suppose z is given along a curve C in the (x, y) plane. Through each point A on C, we can continue the solution along Γ_A in both directions provided Γ_A is not parallel to C, i.e. provided that, on C, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is nowhere equal to $\frac{Q}{P}$. The curves Γ_A are known as the characteristics. Provided the characteristics do not intersect, we obtain a region bounded by Γ_+ and Γ_- within which z is known. If $\frac{Q}{P}$ is independent of z the characteristics are independent of the boundary conditions. In particular $\frac{Q}{P}$ is independent of z for a linear PDE. If P and Q are constants, then the characteristics are parallel straight lines.

Example 1

Solve $z_x - z_y = 1$ with $z = x^2$ on $y = 0$.

Solution

The associated equations are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-1} = \frac{\mathrm{d}z}{1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -1, \ \frac{\mathrm{d}z}{\mathrm{d}x} = 1.$$

Thus the characteristics are $x + y = \alpha$ and on the characteristics $z - x = \beta$.

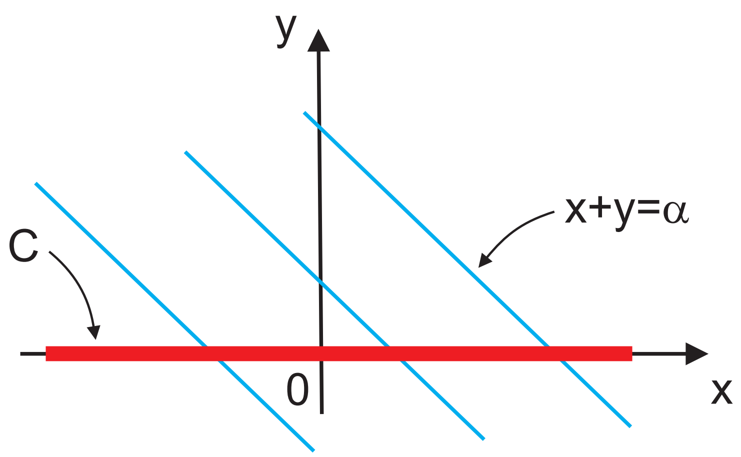

Figure 2: Characteristics of Example 1

The curve C is $y = 0$ and each point on C is intercepted by exactly one characteristic. We can proceed in two ways.

(A) When $y = 0$, $x + y = \alpha \Rightarrow x = \alpha$ and $z = \alpha^2$.

Hence from $z = x + \beta \Rightarrow \beta = \alpha^2 - \alpha$.

Thus $z = x + \left(\alpha^2 - \alpha\right)$ on $x + y = \alpha$.

Eliminate α to get $z = x + \left((x + y)^2 - (x + y)\right) \Rightarrow$

$$z = (x + y)^2 - y.$$

(B) Since $z - x$ is constant when $x + y$ is constant $\Rightarrow$

$z - x = f(x + y)$ for some function f. But $z = x^2$ when $y = 0 \Rightarrow x^2 - x = f(x)$.

Thus $z = x + \left((x + y)^2 - (x + y)\right) \Rightarrow$

$$z = (x + y)^2 - y$$

Example 2

Solve $yz_x + xz_y = z$ with $z = x^3$ on $y = 0$ and $z = y^3$ on $x = 0$.

Solution

The associated equations are

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}z}{z} \Rightarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \Rightarrow x^2 - y^2 = \alpha$$

are characteristics, where α =const. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z}{\sqrt{(x^2-\alpha)}} \Rightarrow \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}x}{\sqrt{(x^2-\alpha)}}$$

$\Rightarrow$

$$\begin{aligned} \ln z &= \ln\left(x + \sqrt{(x^2-\alpha)}\right) + \beta' \\ z &= \beta\left(x + \sqrt{(x^2-\alpha)}\right) \end{aligned}$$

on a characteristic, where $\beta = \mathrm{e}^{\beta'}$ =const.

$$\begin{aligned} \alpha = x^2 - y^2 \Rightarrow z &= \beta\left(x + \sqrt{(x^2 - x^2 + y^2)}\right) \\ &= \underline{\beta(x+y)} \quad \text{or} \quad \underline{\beta(x-y)}. \end{aligned}$$

Case 1: $z = \beta(x + y)$

In this case we find, that as in Ex 1 (B) above,

$$\frac{z}{x+y} \quad \text{is constant when} \quad x^2 - y^2 \quad \text{is constant.}$$

The GS is therefore $z = (x + y)f\left(x^2 - y^2\right)$, and it remains to determine f.

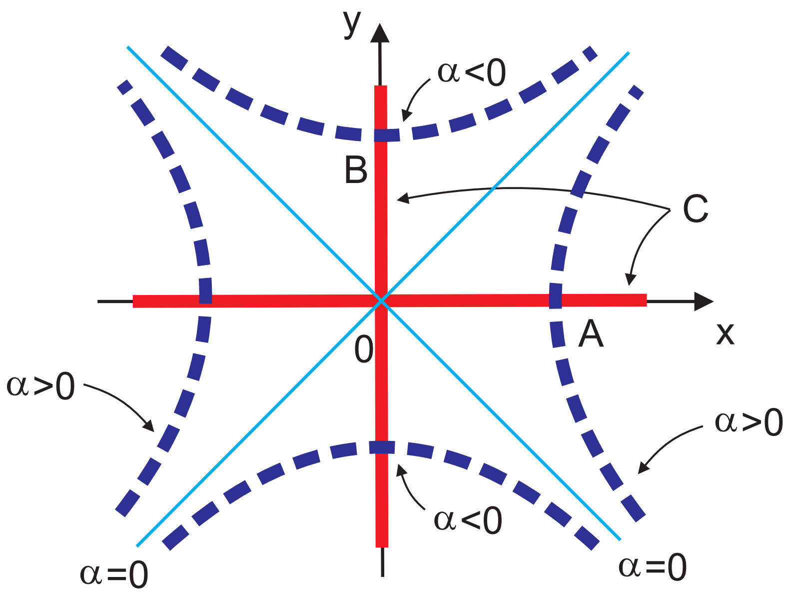

Figure 3: Characteristics of Example 2

We are given z on both axes. At the common point O, $z = 0$ from both prescriptions. The characteristics through O are $x = \pm y$ $(\alpha = 0)$ and on these $z = 0$. The result is obviously symmetric about both axes.

Suppose $\alpha > 0$.

Consider $A(\alpha_1^{\frac{1}{2}}, 0)$ at which

$$z = x^3 = \alpha_1^{\frac{3}{2}}.$$

So from the GS $\Rightarrow$

$$\alpha_1^{\frac{3}{2}} = \alpha_1^{\frac{1}{2}} f(\alpha_1) \Rightarrow f(\alpha_1) = \alpha_1 \Rightarrow z = (x+y)(x^2 - y^2)$$

for $x^2 > y^2$.

Suppose $\alpha < 0$.

Consider $B(0, (-\alpha_2)^{\frac{1}{2}})$ at which

$$z = y^3 = (-\alpha_2)^{\frac{3}{2}}.$$

So from the GS $\Rightarrow$

$$\begin{aligned} (-\alpha_2)^{\frac{3}{2}} &= (-\alpha_2)^{\frac{1}{2}}) f(\alpha_2) \Rightarrow f(\alpha_2) = -\alpha_2 \\ \Rightarrow z &= (x+y)(y^2 - x^2) \quad \text{for} \quad x^2 < y^2 \end{aligned}$$

In summary

$$z = \begin{cases} (x+y)(x^2 - y^2) & \text{for} \quad x^2 > y^2 \\ 0 & \text{for} \quad x^2 = y^2 \\ (x+y)\left(y^2 - x^2\right) & \text{for} \quad x^2 < y^2 \end{cases} \tag{5}$$

Case 2: $z = \beta(x - y)$

In this case it can be similarly deduced that the GS of the PDE is

$$z = (x - y)g(x^2 - y^2).$$

However, this GS gives

$$\begin{aligned} z_x &= g\left(x^2 - y^2\right) + 2x(x - y)g'\left(x^2 - y^2\right) \\ z_y &= -g\left(x^2 - y^2\right) - 2y(x - y)g'\left(x^2 - y^2\right) \end{aligned}$$

Therefore

$$yz_x + xz_y = -(x - y)g(x^2 - y^2) = -z$$

which is not our original PDE, therefore we dismiss this second case, $z = \beta(x - y)$, as spurious solution.

5.2 Some properties of characteristics

• We begin by considering Eq. (5). It is clear that z is everywhere continuous, and that z_x, z_y are everywhere continuous except possibly on the lines

$$x = \pm y \qquad (x^2 - y^2) = 0.$$

From Eq. (5) we find

$$\begin{aligned} x^2 > y^2 : \;\; z_x &= (x^2 - y^2) + 2x(x+y) = (x+y)(3x-y) \\ z_y &= (x^2 - y^2) - 2y(x+y) = (x+y)(x-3y) \end{aligned}$$

$$\begin{aligned} x^2 < y^2 : \;\; z_x &= (y^2 - x^2) - 2x(x+y) = (x+y)(y-3x) \\ z_y &= (y^2 - x^2) + 2y(x+y) = (x+y)(3y-x). \end{aligned}$$

Thus as $x \longrightarrow -y$ from either side, $z_x \longrightarrow 0$ and $z_y \longrightarrow 0$. Hence z_x and z_y are continuous on $x + y = 0$.

However, as $x \longrightarrow y$, z_x jumps from $+4x^2$ $(x > y)$ to $-4x^2$ $(x < y)$, and z_y jumps from $-4x^2$ $(x > y)$ to $+4x^2$ $(x < y)$. Thus z_x and z_y are discontinuous across the characteristic $x = y$.

We investigate the possibility of discontinuities in z_x and z_y for Eq.(1), but we shall suppose z is everywhere continuous. (The standard terminology is that we are looking for weak discontinuities whereas discontinuities in z itself are strong discontinuities).

Suppose C is approached from $+$ and $-$. Then

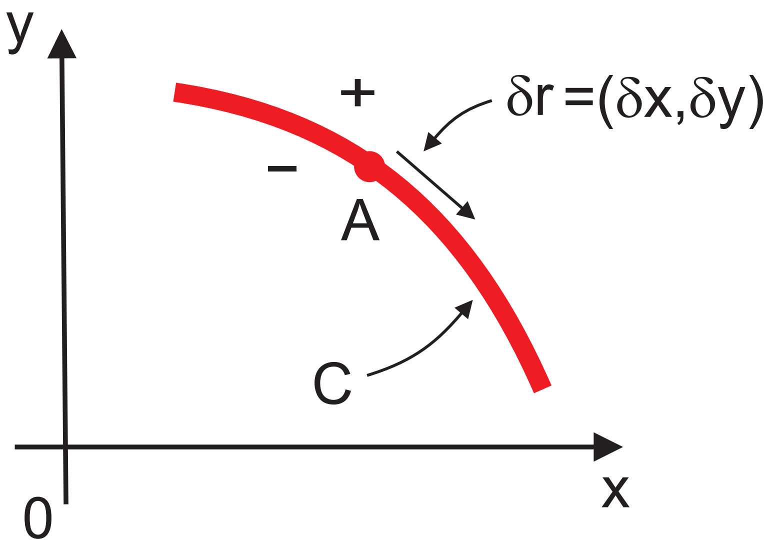

Figure 4: Jump across characteristics

$$\delta z^+ = \frac{\partial z^+}{\partial x}\delta x + \frac{\partial z^+}{\partial y}\delta y$$
$$\delta z^- = \frac{\partial z^-}{\partial x}\delta x + \frac{\partial z^-}{\partial y}\delta y$$

where $(\delta x, \delta y)$ is along C. Subtract.

Because z is continuous, $\delta z^+ = \delta z^-$. Hence

$$\delta x \left[\frac{\partial z}{\partial x}\right]_-^+ + \delta y \left[\frac{\partial z}{\partial y}\right]_-^+ = 0. \tag{6}$$

where the square brackets denote the jump in the expression across C.

Since Eq. (1) is satisfied on both sides and since, by hypothesis, P, Q, R are continuous

$$P \left[\frac{\partial z}{\partial x}\right]_-^+ + Q \left[\frac{\partial z}{\partial y}\right]_-^+ = 0. \tag{7}$$

The necessary condition for

$$\left[\frac{\partial z}{\partial x}\right]_-^+ \neq 0 \quad \text{and} \quad \left[\frac{\partial z}{\partial y}\right]_-^+ \neq 0 \quad \text{is} \quad \frac{\delta x}{P} = \frac{\delta y}{Q},$$

i.e.

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}. \tag{8}$$

Thus C must be a characteristic Γ_A.

• When Q/P is independent of z, the characteristics are independent of the boundary conditions.

When Q/P depends on z, different boundary conditions produce different sets of characteristics.

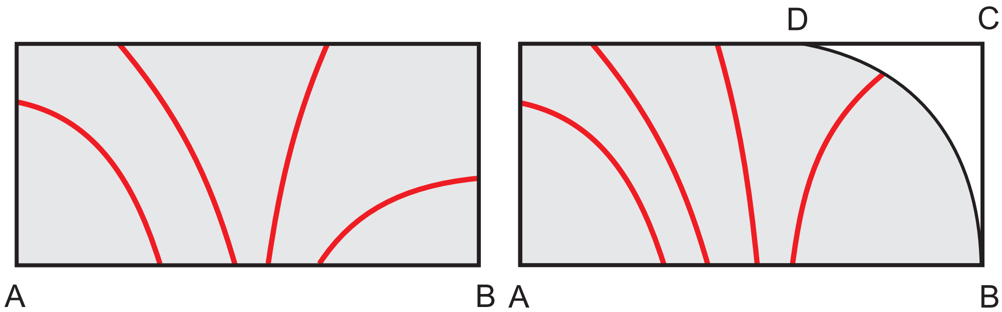

Figure 5: (i) z determined throughout rectangle; (ii) z not determined in BCD

We can also consider situations in which z itself is discontinuous at a point on the boundary. Then the shape of the characteristics (in the case when Q/P depends on z) will change discontinuously at that point. Qualitatively, there are two possibilities:

- In (i) there appear to be two characteristics through a point, whereas
- in (ii) there is a region containing no characteristics.

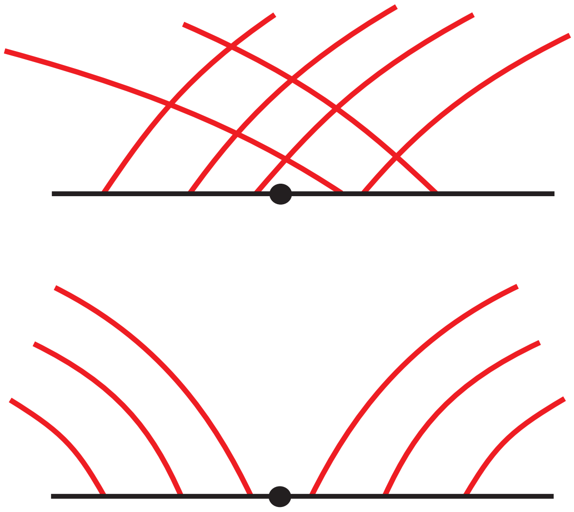

Figure 6: Characteristics leading to (i) shocks; or to (ii) centred fans (also called rarefication shocks)

Again, qualitatively these two situations will be relevant to our models of traffic flow [(i) leads to shocks, (ii) leads to centred fans - see later].

• We can obtain similar situations to (i) and (ii), but even without initial discontinuities in the slopes of the characteristics. The following example will connect well with our models of traffic flow.

Example

Solve

$$\rho_t + \rho\rho_x = 0$$

with $\rho = f(x)$ on $t = 0$. Consider two special cases:

Case 1 : $f(x) = 0 \quad (x < 0), \quad f(x) = x \quad (0 \leq x < 1),$
$f(x) = 1 \quad (x \geq 1);$

Case 2 : $f(x) = 0 \quad (x < 0), \quad f(x) = -x \quad (0 \leq x < 1),$
$f(x) = -1 \quad (x \geq 1).$

The associated equations Eq. (4) are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\rho} = \frac{\mathrm{d}\rho}{0}$$

where the last is to be interpreted as $\mathrm{d}\rho = 0$.

Thus

$$\rho = \alpha, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \alpha \Rightarrow x - \alpha t = \beta \quad \text{are characteristics.}$$

Now consider the characteristic through $x = \xi$ on $t = 0$. On this characteristic $\rho = \alpha = f(\xi)$. Thus

$$x = f(\xi)t + \xi, \;\; \rho = f(\xi) \tag{9}$$

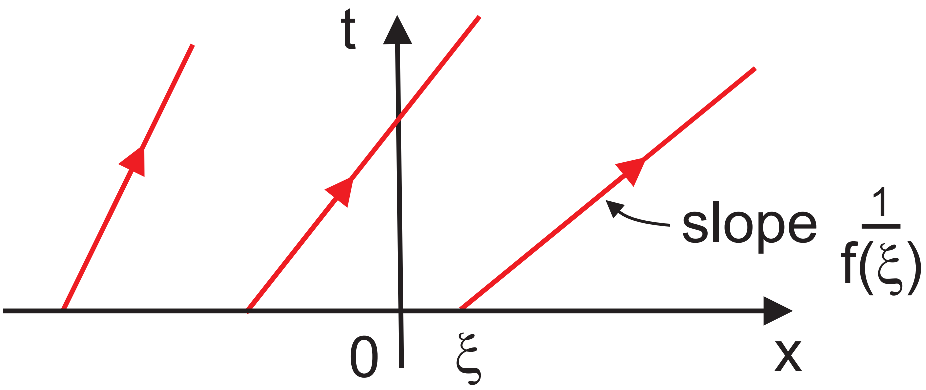

Figure 7: Characteristics for Example

Alternatively, from Eq. (9)

$$x = \rho t + \xi$$

so that we can write Eq. (9) implicitly as

$$\rho = f(x - \rho t) \tag{10}$$

Case 1: From Eq. (10)

$$\rho = 0 \quad (x < 0), \qquad \rho = x - \rho t \quad (0 \leq x - \rho t < 1)$$

$\Rightarrow$

$$\rho = \frac{x}{1+t} \quad (0 \leq x < 1+t), \qquad \rho = 1 \quad (x \geq 1+t),$$

i.e.

$$\rho = \begin{cases} 0 & (x < 0) \\ x/(1+t) & (0 \leq x < 1+t) \\ 1 & (x \geq 1+t) \end{cases} \tag{11a}$$

Case 2: Likewise,

$$\rho = \begin{cases} 0 & (x < 0) \\ -x/(1-t) & (0 \leq x < 1-t) \\ -1 & (x \geq 1-t) \end{cases} \tag{11b}$$

The solution Eq. (11b) breaks down at $t = 1$; as the sketch on the hand-out shows, t he characteristics intersect at $t = 1$ in Case 2 and the profile of ρ against x becomes triple-valued (but this cannot occur in reality).

5.3 Model of traffic flow

We assume:

1. One lane of traffic in direction of Ox with no overtaking.
2. We can define a local car density $\rho = \rho(x, t)$ as the number of cars per unit length of road.
3. The local car velocity $v(x, t)$ is a function of ρ alone, i.e.

$$v = v(\rho) \tag{12}$$

The meaning of Eq. (12) is that each driver adjusts his, her or its speed to local conditions exclusively, whereas most drivers look ahead and adjust speed where appropriate. These assumptions give a car flowrate $q(\rho)$ with

$$q(\rho) = \rho v(\rho) \tag{13}$$

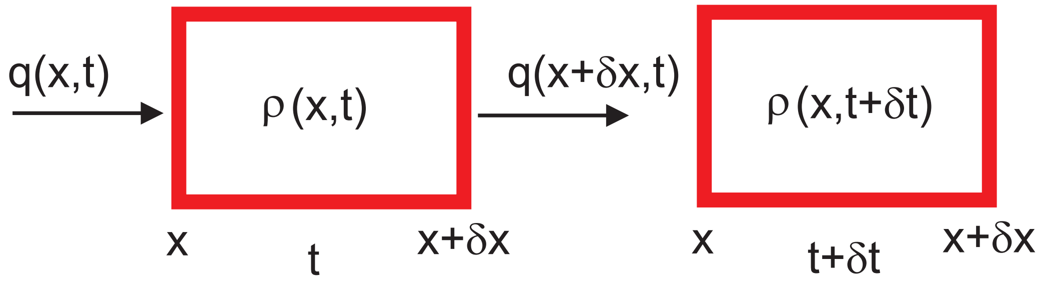

Figure 8: Car flow

Consider two values of x, viz. x_1, x_2 with $x_1 \leq x \leq x_2$.

At time t, the number of cars in this interval is

$$\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x.$$

The rate of change of this must be the net flowrate, viz.

$$\frac{\partial}{\partial t}\left\{\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x\right\} = [q(x,t)]_{x_2}^{x_1} \tag{14}$$

If $x_1 = x$, $x_2 = x + \delta x$, Eq. (14) becomes

$$\frac{\partial}{\partial t}\rho\delta x = -\frac{\partial q}{\partial x}\delta x$$

$\Rightarrow$

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{15}$$

- We need to model $v(\rho)$.

We assume there is a maximum possible density P with essentially "bumper-to-bumper" traffic. When $\rho = P$, we assume $v(\rho)$ in Eq. (12) is zero.

We also assume $v(\rho)$ decreases as ρ increases, with a maximum of V when $\rho = 0$.

These assumptions are shown schematically...

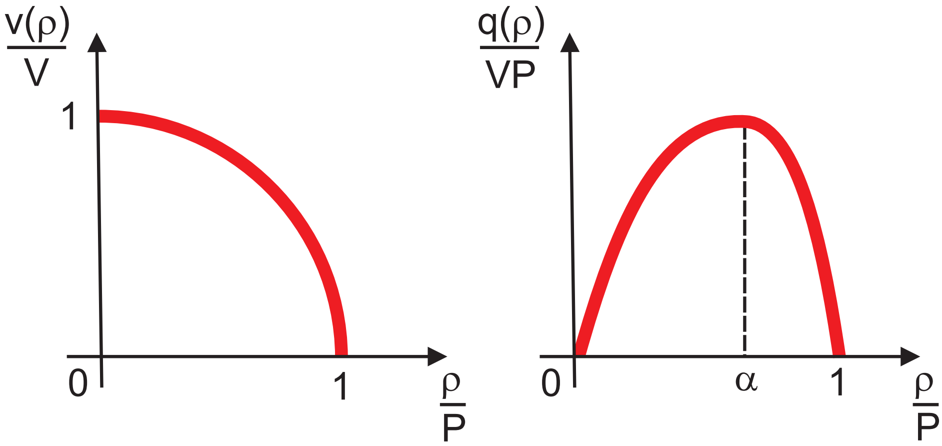

Figure 9: Modelling car flow

With Eq. (13), Eq. (15) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho v(\rho)\right)\frac{\partial \rho}{\partial x} = 0$$

or

$$\begin{aligned} \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} &= 0, \\ c(\rho) &= \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho v(\rho)\right) = v(\rho) + \rho v'(\rho) \end{aligned} \tag{16}$$

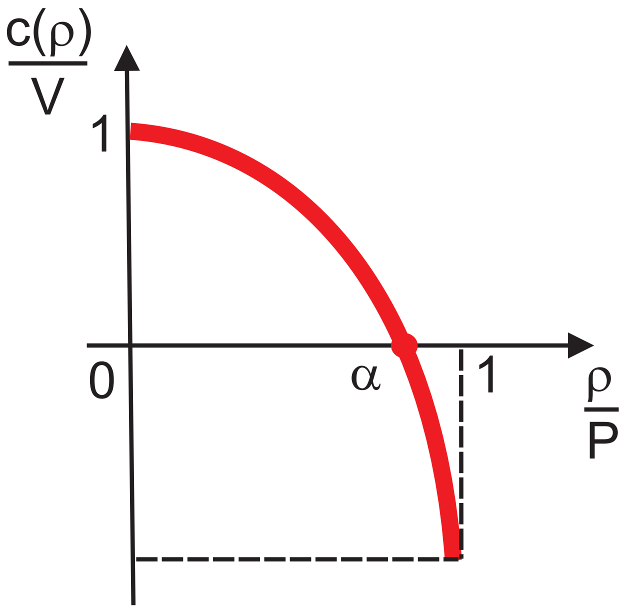

Figure 10: Model of car flow

The assumptions made about $v(\rho)$ give a $c(\rho)$ which is monotonic decreasing and negative for $\rho/P > \alpha$, where α is the value of ρ/P for which $q(\rho)$ is a maximum.

5.4 Small amplitude disturbances from a uniform state

• Before studying the full non-linear problem, it is instructive to consider a simpler one. Suppose that there is almost a uniform state with $\rho = \rho_0$ and

$$\rho = \rho_0 + \rho' \text{ with } |\rho'| \ll \rho_0. \tag{17}$$

Linearise Eq. (16) - as with sound waves earlier - to get

$$\frac{\partial \rho'}{\partial t} + c\,(\rho_0)\,\frac{\partial \rho'}{\partial x} = 0. \tag{18}$$

Either

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c\,(\rho_0)} = \frac{\mathrm{d}\rho'}{0} \Rightarrow$$

$$\rho' = \text{const.} \quad \text{on} \quad x - c(\rho_0)t = \text{const.}$$

Or put $\xi = x - c\,(\rho_0)\,t \Rightarrow$

$$\frac{\partial \rho'}{\partial x} = \frac{\partial \rho'}{\partial \xi}, \quad \left(\frac{\partial \rho'}{\partial t}\right)_x = \left(\frac{\partial \rho'}{\partial t}\right)_\xi - c\,(\rho_0)\left(\frac{\partial \rho'}{\partial \xi}\right) \Rightarrow$$

$$\left(\frac{\partial \rho'}{\partial t}\right)_\xi = 0.$$

Thus the GS of Eq. (18) is

$$\rho' = f\,\{x - c\,(\rho_0)\,t\}. \tag{19}$$

- The characteristics of Eq. (18) are the straight lines

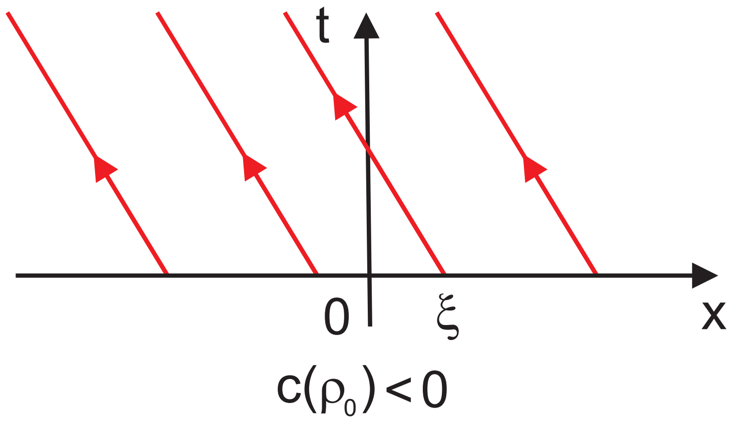

Figure 11: Characteristics of car flow are straight lines

$$x = \xi + c\,(\rho_0)\,t. \tag{20}$$

Eq. (19) shows that ρ' is constant on each characteristic.

Eq. (19) represents a wave travelling to the right with speed $c\,(\rho_0)$. If $\rho_0/P > \alpha \Rightarrow c(\rho_0) < 0$.

This is a kinematic wave; $c\,(\rho_0)$ is the speed of the disturbance, not of the cars.

This explains a common phenomenon on a busy road when a sudden increase in density reaches you from ahead with no apparent reason.

5.5 The initial value problem for Eq. (16)

- We wish to solve Eq. (16) subject to the initial condition

$$\rho(x, 0) = f(x) \tag{21}$$

By the earlier methods - see especially the Example in § (5.2) - ρ is constant on the characteristics

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c(\rho)} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = c(\rho).$$

Since ρ is constant on a characteristic, the characteristics are straight.

$\Rightarrow$

Thus, if $c\{f(\xi)\} = F(\xi)$, the solution can be written for $t \geq 0$

$$\rho = f(\xi) \quad \text{on the straight line} \quad x = \xi + F(\xi)t. \tag{22}$$

Example

Suppose

$$v(\rho) = \frac{V}{P}(P - \rho) \tag{23}$$

and that $\rho(x, 0) = f(x)$ satisfies

$$\rho = \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(\rho_L - \rho_R)\tanh\frac{x}{L} \tag{24}$$

where ρ_L, ρ_R and L are constants. Discuss the solution given by Eq. (22) when

$$(i) \quad \rho_L > \rho_R, \quad \text{and} \quad (ii) \quad \rho_L < \rho_R.$$

Solution

From flow model Eq. (23) it follows

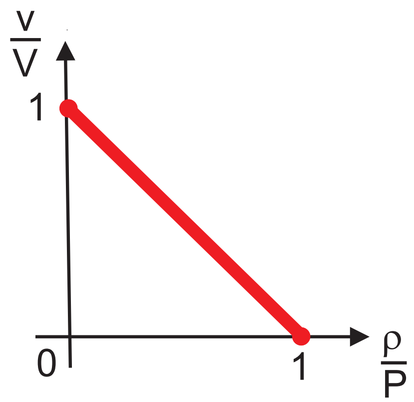

Figure 12: (a) Car flow $v(\rho) = V(P - \rho)/P$

$$q(\rho) = \frac{V}{P}\left(P\rho - \rho^2\right), \quad c(\rho) = \frac{V}{P}(P - 2\rho). \tag{25}$$

Note also that

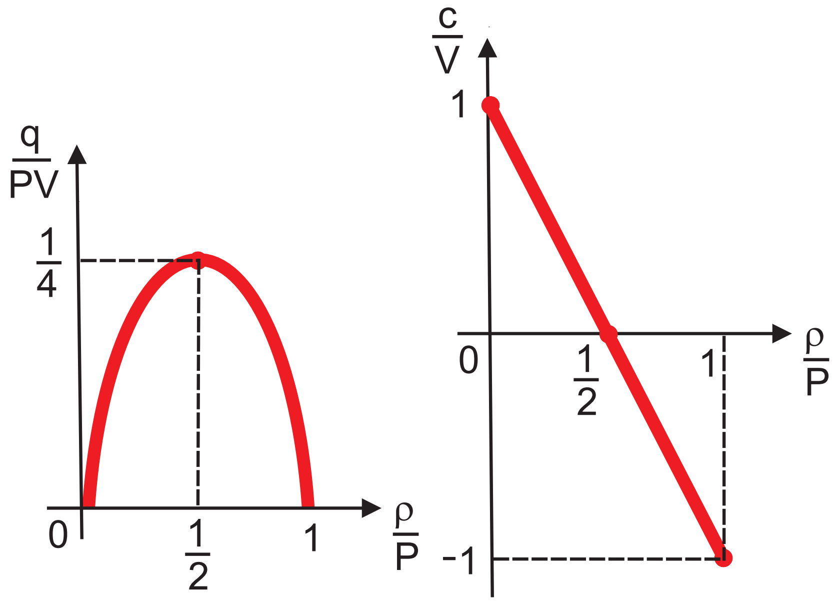

Figure 12: (b) Car flow flux, and (c) speed of disturbance

$$\rho \to \rho_L \quad \text{as} \quad \frac{x}{L} \to -\infty$$

and

$$\rho \to \rho_R \quad \text{as} \quad \frac{x}{L} \to +\infty.$$

Also

$$\begin{aligned} F(\xi) &= c\{f(\xi)\} \qquad (26) \\ &= \frac{V}{P}\left[P - \rho_L - \rho_R + (\rho_L - \rho_R)\tanh\left(\frac{\xi}{L}\right)\right] \end{aligned}$$

Case (i): $\rho_L > \rho_R \Rightarrow F'(\xi) > 0 \quad \text{for} \quad \forall \xi$

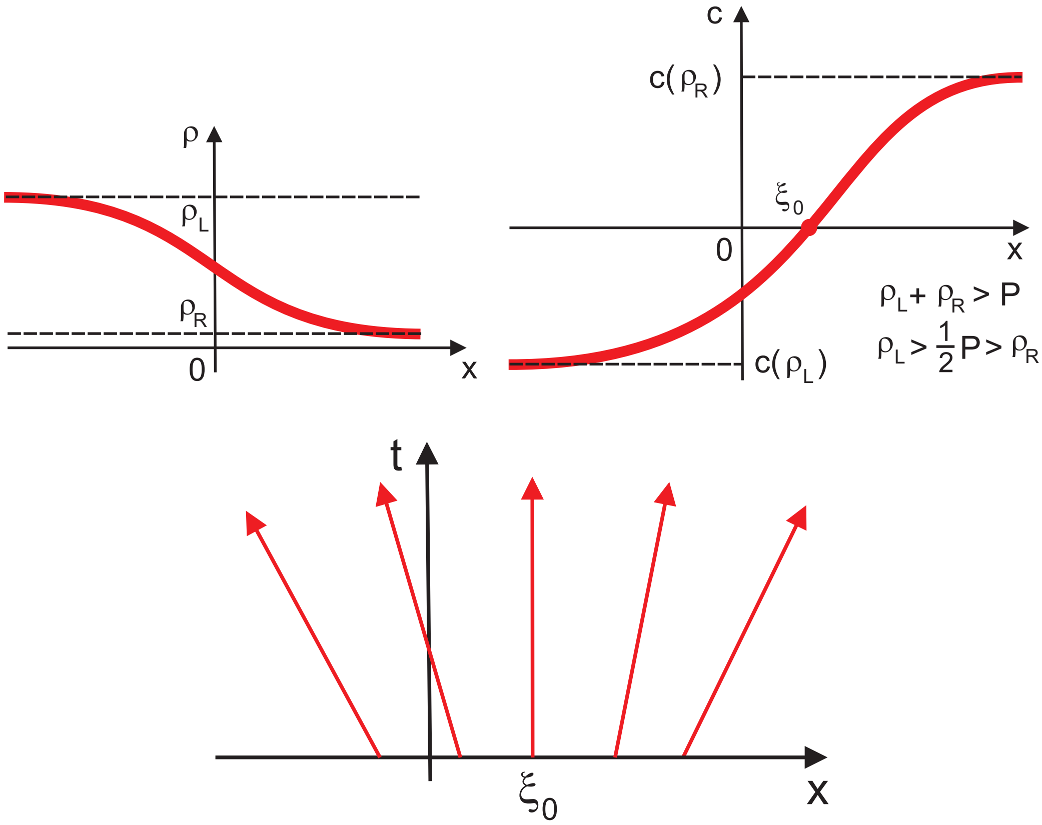

Figure 13: (a) Car flow, (b) profile of speed of disturbance, and (c) characteristics for Case (i). Note, that ρ is constant on each characteristic

Case (ii): $\rho_L < \rho_R \Rightarrow F'(\xi) < 0 \quad \text{for} \quad \forall \xi$

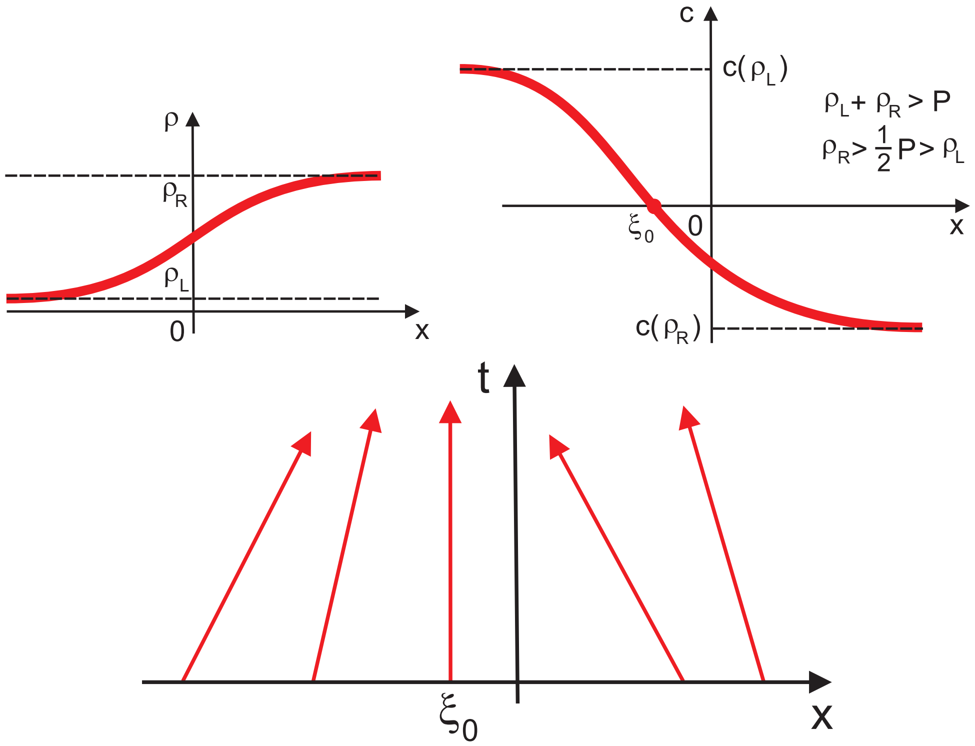

Figure 14: (a) Car flow, (b) profile of speed of disturbance, and (c) characteristics for Case (ii). Note, from (c) that characteristics eventually intersect

Characteristics eventually intersect $\Rightarrow$ problem becomes ill-posed.

If two characteristics intersect, any enclosed characteristic must meet one of them at an earlier time $\Rightarrow$ earliest intersection must be between neighbouring characteristics.

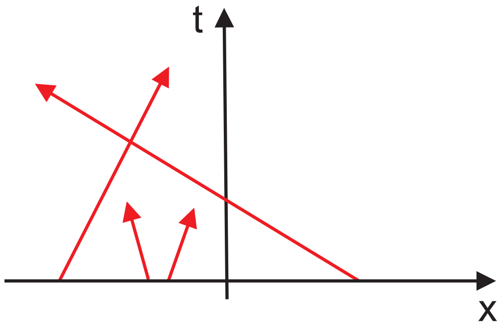

Figure 15: Intersecting characteristics

Suppose these are

$$\left.\begin{array}{rcl} x &=& \{\xi + F(\xi)t\} \\ \\ x &=& \{(\xi + \partial\xi) + F(\xi + \partial\xi)t\} \\ &=& \{\xi + F(\xi)t\} + \{1 + F'(\xi)t\}\,\partial\xi \end{array}\right\} \Rightarrow$$

$$\therefore 1 + F'(\xi)t = 0. \tag{27}$$

We get solutions of Eq. (27) with $t > 0$ only if $\exists\,\xi$ with $F'(\xi) < 0$. [Thus for $\rho_L > \rho_R$ there are no intersections and the solution given by Eq. (22) applies for $\forall\ t \geq 0$.]

The first positive t satisfying Eq. (27) occurs when

$$t = T_{\min} = \frac{1}{\underset{-\infty<\xi<\infty}{\text{Max}} \{-F'(\xi)\}}. \tag{28}$$

While Eqs. (27) and (28) are general, we can calculate $T_{\min}$ in our particular case when Eq. (26) holds.

We find

$$-F'(\xi) = \frac{V}{P}(\rho_R - \rho_L)\frac{1}{L}\text{sech}^2\left(\frac{\xi}{L}\right)$$

$\Rightarrow$

$$\max\{-F'(\xi)\} = \frac{V(\rho_R - \rho_L)}{PL}$$

when $\xi = 0$. Then Eq. (28) gives

$$T_{\min} = \frac{PL}{V(\rho_R - \rho_L)}. \tag{29}$$

5.6 Shocks

• We can understand in another way why there is trouble when $\rho_L < \rho_R$. With $c'(\rho) < 0$, low densities propagate forward relative to high densities. The profile of ρ against x inevitably steepens as t increases and has a vertical section at $t = T_{\min}$. Were we to continue, the profile would develop the triple-valued shape - clearly unacceptable since ρ must be a single valued quantity.

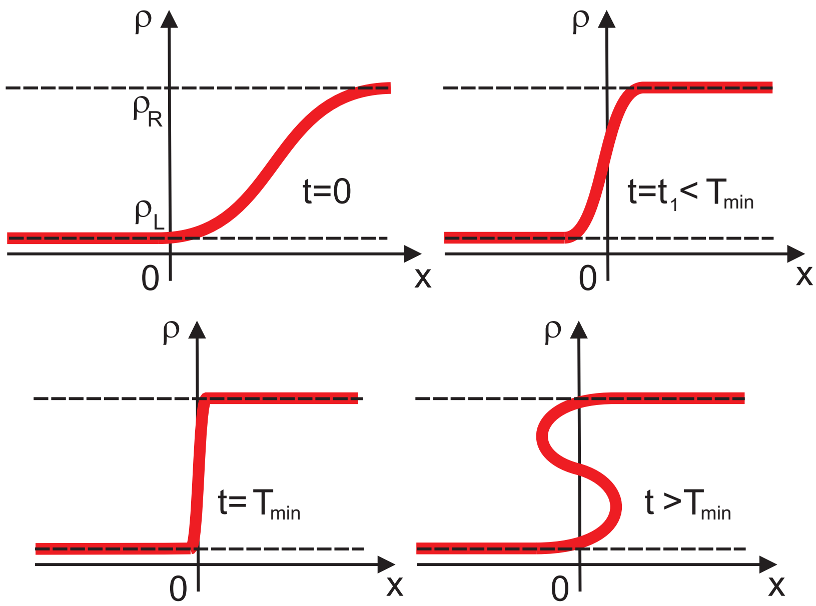

Figure 16: Development of shock

• Instead the wave breaks, and the model must be extended. A consistent extension conserves cars but allows discontinuities in ρ to occur across a shock.

We cannot have characteristics crossing one another. Instead the picture is as shown schematically.

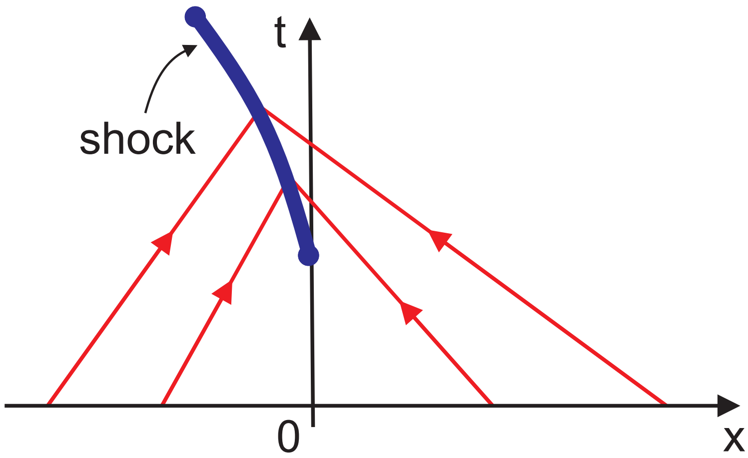

Figure 17: Shock front and characteristics

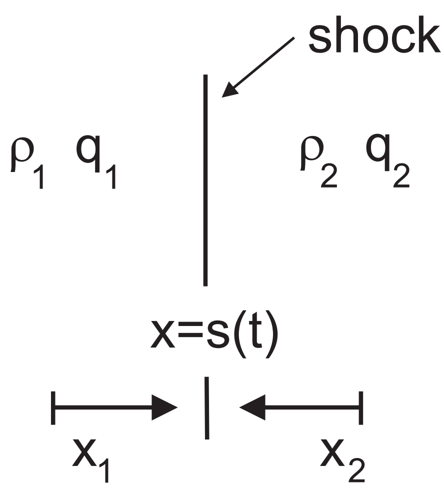

Figure 18: Quantities at a shock front

From Eq. (14) $\Rightarrow$

$$\frac{\partial}{\partial t}\int_{x_1}^{s(t)} \rho \mathrm{d}x + \frac{\partial}{\partial t}\int_{s(t)}^{x_2} \rho \mathrm{d}x = q_1 - q_2$$

$$\begin{aligned}
\text{LHS} &= \underbrace{\int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t}\mathrm{d}x}_{\to 0 \text{ as } x_1 \to s_-} + \frac{1}{\partial t}\left\{\int_{x_1}^{s(t+\delta t)} - \int_{x_1}^{s(t)}\right\}\rho \mathrm{d}x \\
&\quad + \underbrace{\int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t}\mathrm{d}x}_{\to 0 \text{ as } x_2 \to s_+} + \frac{1}{\partial t}\left\{\int_{s(t+\delta t)}^{x_2} - \int_{s(t)}^{x_2}\right\}\rho \mathrm{d}x \\
&= \frac{1}{\partial t}\left\{\int_{s(t)}^{s(t)+\delta t} \rho_1 \mathrm{d}x - \int_{s(t)}^{s(t)+\delta t} \rho_2 \mathrm{d}x\right\} \\
&= \dot{s}\left(\rho_1 - \rho_2\right)
\end{aligned}$$

by the mean value theorem.

$\Rightarrow$

$$\dot{s}\left(\rho_1 - \rho_2\right) = \left(q_1 - q_2\right) \tag{30a}$$

$$\dot{s} = \frac{q_1 - q_2}{\rho_1 - \rho_2}. \tag{30b}$$

• The position of the shock is fixed by the need to conserve cars. Without a shock the curve of ρ against x would become (unacceptably) triple-valued. We insert the shock so that the shaded areas are equal, thus ensuring that $\int \rho \mathrm{d}x$ = number of cars is unchanged. This is known as (Whitham's) equal area rule.

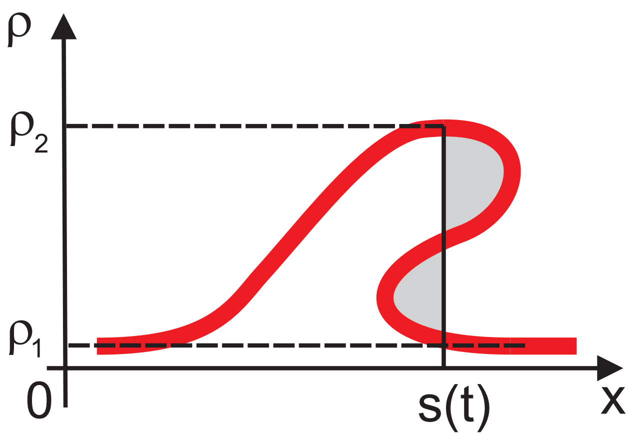

Figure 19: Shock fitting: Whitham's equal area rule

• As an application consider what happens as cars approach a stationary queue behind a red traffic light so that $\rho_R = P$, $\rho_L < P$. On meeting the queue cars stop and the lengthening of the queue is achieved by a shock wave propagating backwards.

5.7 The Riemann problem

- We wish to consider the case when we solve Eqs. (16) and (21) where there is a discontinuity in $f(x)$. It will be sufficient to consider the simplest possible case, viz.

$$\rho(x,0) = f(x) = \begin{cases} \rho_L & (x < 0) \\ \\ \rho_R & (x > 0) \end{cases} \tag{31}$$

- Then Eq. (22) gives the solution as

$$\begin{aligned} \rho = \rho_L \quad \text{on } x = \xi + c(\rho_L)\,t \quad (\xi < 0) \\ \\ \rho = \rho_R \quad \text{on } x = \xi + c(\rho_R)\,t \quad (\xi > 0) \end{aligned} \tag{32}$$

Consider first the case $\rho_L > \rho_R$. The characteristic diagram is easy to draw...

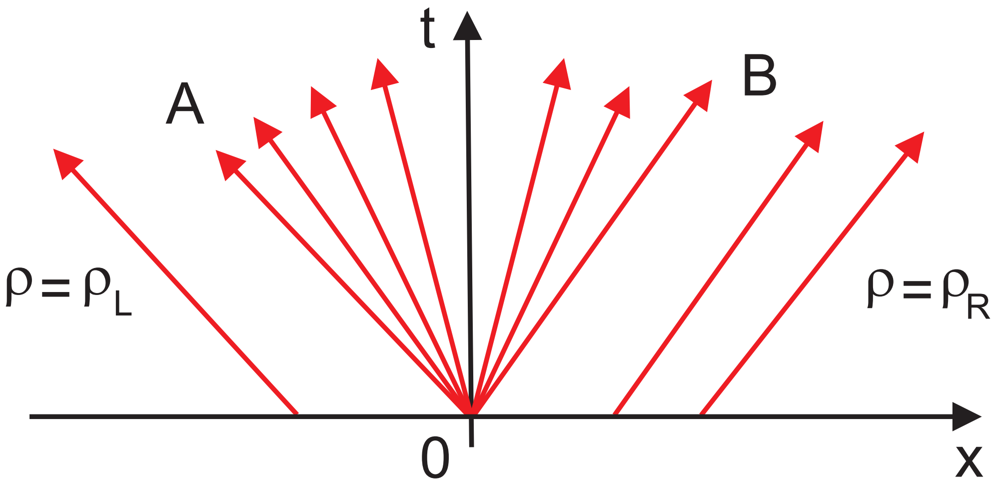

Figure 20: Fan of characteristics

They are either parallel to OA with slope $c\,(\rho_L)$; to the left of OA, $\rho = \rho_L$. Or they are parallel to OB with slope $c(\rho_R)$; to the right of OB, $\rho = \rho_L$. But what happens in OAB?

• The problem arises because of the discontinuity and can be solved by considering a limit process in which ρ takes all the values from ρ_R to ρ_L, and all the characteristics go through the origin. Thus

$$\rho = k \qquad (\rho_R < \rho < \rho_L)$$

on $x = c(k)t$.

The solution is therefore:

$$\rho = \begin{cases} \rho_L & : \quad x < c\,(\rho_L)\,t \\ k \text{ on } x = c(k)t & : \quad c\,(\rho_L) < c(k) < c\,(\rho_R) \\ \rho_R & : \quad x > c\,(\rho_R)\,t \end{cases} \quad (33)$$

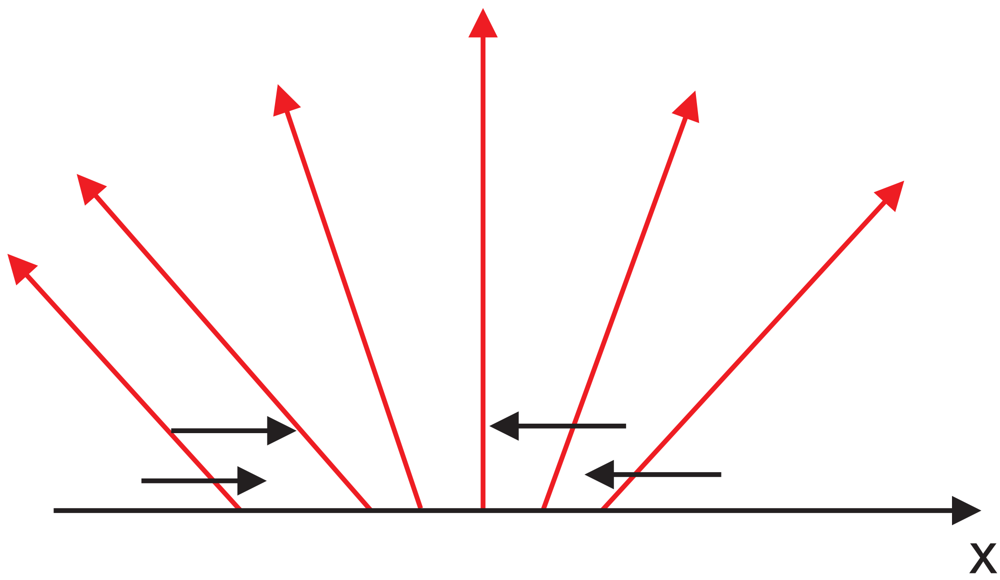

Figure 21: Centered fan or expansion fan corresponding to rarefication wave

The characteristic diagram is augmented by a centred fan or an expansion fan or expansion wave or rarefication wave.

• Conversely, when $\rho_L < \rho_R$ the characteristic diagram shows immediately trouble whose only resolution is a shock starting from $t = 0$ with speed, given by Eq. (30b) as U, where

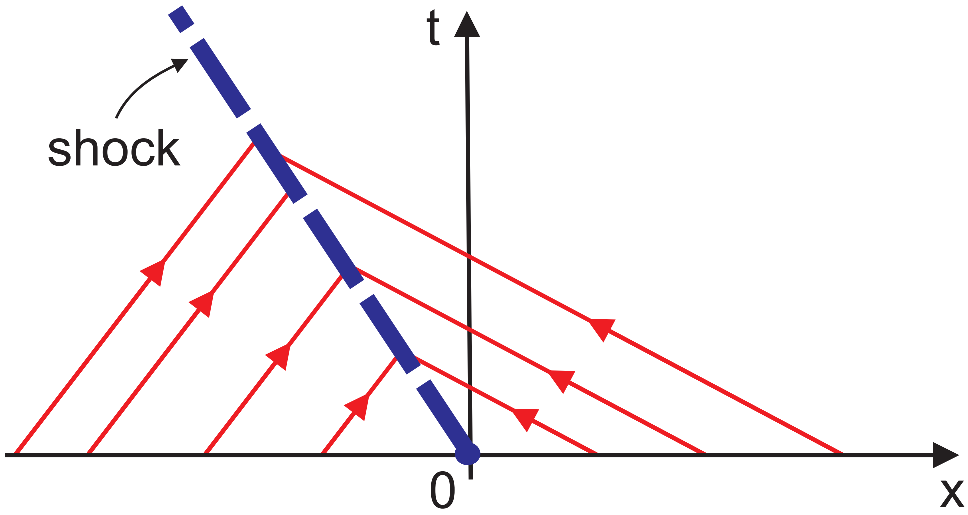

Figure 22: Schematic shock with speed U

$$U = \frac{q\left(\rho_L\right) - q\left(\rho_R\right)}{\left(\rho_L - \rho_R\right)} \tag{34}$$

5.8 Additional refinements

• The model assumptions leading to Eq. (16) are too simple. One extension is to suppose that q is a function of the density gradient $\partial\rho/\partial x$ as well as ρ, thus allowing drivers to reduce their speed to account for an increasing density ahead. A simple assumption is to take

$$q = Q(\rho) - \nu\rho_x \tag{35}$$

where ν is a positive constant. Thus q decreases if ρ_x is positive, i.e. it there is an increasing density ahead. Use of Eq. (35) in Eq. (15) gives

$$\rho_L + c(\rho)\rho_x = \nu\rho_{xx}, \;\; c(\rho) = q'(\rho) \tag{36}$$

• Seek solutions of Eq. (36) of the form

$$\rho = \rho(X) \qquad X = x - Ut \tag{37}$$

where U is a constant still to be determined. Substitution in Eq. (36) $\Rightarrow$

$$-U\rho'(X) + c(\rho)\rho'(X) = \nu\rho''(X).$$

Since $c(\rho) = Q'(\rho)$ we have

$$Q(\rho) - U\rho + C = \nu\rho'(X) \tag{38}$$

where C is a constant. Suppose $\rho \to \rho_L$ as $X \to -\infty$ and $\rho \to \rho_R$ as $X \to +\infty \Rightarrow$

$$Q(\rho_L) - U\rho_L + C = Q(\rho_R) - U\rho_R + C = 0,$$

$\Rightarrow$

$$U = \frac{Q(\rho_R) - Q(\rho_L)}{\rho_R - \rho_L} \tag{39}$$

This is exactly Eq. (30b) but (for the moment) in a different context.

• Since $\rho \to \rho_L$ as $X \to -\infty$ and $\rho \to \rho_R$ as $X \to +\infty$, $\rho'(x) = 0$ at $\rho = \rho_L$ and $\rho = \rho_R$. We suppose ρ_L and ρ_R are simple zero's of

$$Q(\rho) - U\rho + C,$$

and more precisely we shall suppose $\rho_L < \rho_R$ and

$$Q(\rho) - U\rho + C = \alpha(\rho - \rho_L)(\rho_R - \rho) \qquad (\alpha > 0) \tag{40}$$

With $\alpha > 0$,

$$c(\rho) = Q'(\rho) = \alpha\,(\rho_R - \rho) - \alpha\,(\rho - \rho_L)$$

and

$$c'(\rho) = \alpha\,(\rho_L - \rho_R) < 0.$$

We can always approximate $Q(\rho)$ by a quadratic. Then Eq. (38) becomes

$$\nu \frac{\mathrm{d}\rho}{\mathrm{d}X} = \alpha\,(\rho - \rho_L)\,(\rho_R - \rho)$$

with solution

$$\left(\frac{\rho_R - \rho}{\rho - \rho_L}\right) = \left(\frac{\rho_R - \rho_0}{\rho_0 - \rho_L}\right) \mathrm{e}^{-\frac{X}{L}}, \qquad L = \frac{\nu}{\alpha\,(\rho_R - \rho_L)} \tag{41}$$

where $\rho = \rho_0$ at $X = 0$. We note that $\rho \to \rho_L$ as $X \to -\infty$ and that $\rho \to \rho_R$ as $X \to +\infty$ as required.

The transition between $\rho \sim \rho_L$ and $\rho \sim \rho_R$ occupies a thickness of order L.

As L diminishes, i.e. as ν diminishes for fixed α and $(\rho_R - \rho_L)$ the transition takes place more sharply $\Rightarrow$ a shock is approached.

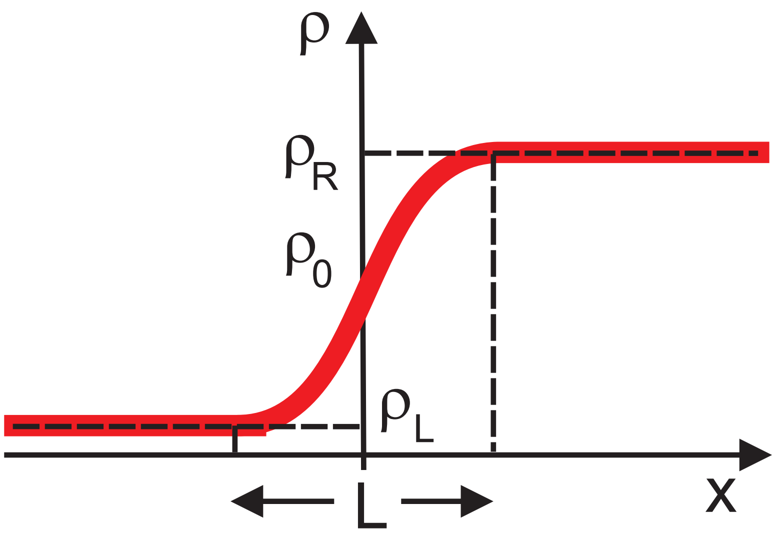

Figure 23: Development of shock front

• The model in this section can be taken further in the case when Eq. (40) holds.

Multiply Eq. (36) by $c'(\rho) \Rightarrow$

$$c'(\rho)\rho_t + c(\rho)c'(\rho)\rho_x = \nu c'(\rho)\rho_{xx}$$

$$\therefore \qquad c_t + cc_x = \nu\tfrac{\partial^2 c}{\partial x^2} - \nu c''(\rho)\left(\tfrac{\partial \rho}{\partial x}\right)^2$$

because

$$\frac{\partial c}{\partial x} = c'(\rho)\frac{\partial \rho}{\partial x}$$

and therefore

$$\frac{\partial^2 c}{\partial x^2} = c''(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2 + c'(\rho)\frac{\partial^2 \rho}{\partial x^2}.$$

In the case when $Q(\rho)$ is quadratic, i.e. Eq. (40) holds, $c''(\rho) = 0$ since $c(\rho) = Q'(\rho)$. Thus

$$c_t + cc_x = \nu c_{xx}. \tag{42}$$

This is known as Burger's equation and, remarkably, it can be solved explicitly by means of the transformation

$$c = -2\nu\frac{\phi_x}{\phi} \tag{43}$$

discovered independently by E. Hopf (1950) and J.D.Cole (1951). Use of Eq. (43) transforms Eq. (42) into the standard linear equation (after one integration w.r.t. x):

$$\phi_t = \nu\phi_{xx}. \tag{44}$$

It can be shown that this is also consistent with the shock structure.

• A second refinement is that there is a time lag in driver response. One way of handling this is to take Eq. (35) and deduce from it that $v = q/\rho$ satisfies

$$v = V(\rho) - \frac{\nu}{\rho}\rho_x, \qquad V(\rho) = \frac{Q(\rho)}{\rho}. \tag{45}$$

Then regard this as a velocity which the driver tries to achieve. The acceleration of the car is

$$\frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

[see Notes after § (4.3)] and the model is

$$v_t + vv_x = -\frac{1}{\tau}\left\{v - V(\rho) + \frac{\nu}{\rho}\rho_x\right\}, \tag{46}$$

where τ is a measure of the response time. Eq. (46) is to be solved together with Eq. (15), i.e.

$$\rho_t + (\rho v)_x = 0. \tag{47}$$

THE END