The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS Autumn Semester 2013–14

WAVES 2 hours

Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics.

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1 (i) *Derive* d'Alembert's general solution for the one-dimensional wave equation

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}$$

on $-\infty < x < \infty$ for $t \geq 0$.

(13 marks)

(ii) Given that $c = 1$ and at $t = 0$

$$\phi(x, 0) = \sin kx, \qquad \frac{\partial\phi}{\partial t} = -k\cos kx,$$

where k is a constant, find $\phi(x, t)$.

(7 marks)

(iii) Give a physical interpretation of your solution. Further, explain why the solution, subject to the initial conditions in (ii), cannot (or can) be a standing wave.

(5 marks)

2 The vibration of a string evolves according to the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

The boundary conditions are that $u = 0$ at both $x = 0$ and $x = a$, and initially, i.e. at $t = 0$, $\partial u/\partial t = 0$.

(i) Using the *method of separation of variables*, show that this configuration has the general solution

$$u(x, t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi}{a}x \cos\frac{n\pi}{a}ct.$$

(16 marks)

(ii) If initially $u = f(x)$, find an expression for A_n in terms of $f(x)$.

(6 marks)

(iii) Show that each term in the general solution can be expressed in terms of

$$\sin\left[\frac{n\pi}{a}(x + ct)\right] \quad \text{and} \quad \sin\left[\frac{n\pi}{a}(x - ct)\right],$$

and give a brief physical interpretation.

(3 marks)

3 Consider acoustic waves in a closed cuboidal box with sides a_1, a_2, a_3. The origin O of a Cartesian coordinate system (x_1, x_2, x_3) is taken at one corner of the box with the axes parallel to the sides of the box. The velocity potential ϕ satisfies

$$\frac{\partial^2\phi}{\partial {x_1}^2} + \frac{\partial^2\phi}{\partial {x_2}^2} + \frac{\partial^2\phi}{\partial {x_3}^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2},$$

where the constant c is the speed of sound. The boundary conditions on ϕ are:

$$\text{(a)}\ \frac{\partial\phi}{\partial x_1} = 0 \text{ at } x_1 = 0,\ x_1 = a_1; \quad \text{(b)}\ \frac{\partial\phi}{\partial x_2} = 0 \text{ at } x_2 = 0,\ x_2 = a_2;$$

$$\text{(c)}\ \frac{\partial\phi}{\partial x_3} = 0 \text{ at } x_3 = 0,\ x_3 = a_3.$$

(i) Seek separable solutions of the form

$$\phi = X_1(x_1)X_2(x_2)X_3(x_3)\cos\omega t,$$

where ω is a positive constant.

Show that X_i''/X_i must be constant for each i, and that these constants must all be non-positive. Here $'$ denotes differentiation.

Deduce that

$$\phi \propto \cos\left(\frac{n_1\pi x_1}{a_1}\right)\cos\left(\frac{n_2\pi x_2}{a_2}\right)\cos\left(\frac{n_3\pi x_3}{a_3}\right)\cos\omega t,$$

where the non-negative integers n_1, n_2, n_3 satisfy

$$\frac{n_1^2}{A_1^2} + \frac{n_2^2}{A_2^2} + \frac{n_3^2}{A_3^2} = 1 \text{ with } A_1 = \frac{a_1\omega}{\pi c},\ A_2 = \frac{a_2\omega}{\pi c},\ A_3 = \frac{a_3\omega}{\pi c}.$$

(18 marks)

(ii) Deduce that, for a large box, the number of different waves with angular frequency less than or equal to ω is approximately equal to

$$\frac{\omega^3}{6\pi^2c^3}a_1a_2a_3.$$

(7 marks)

[*HINT. Consider the number of integer triples* (n_1, n_2, n_3) *within the surface* $\frac{x_1^2}{A_1^2} + \frac{x_2^2}{A_2^2} + \frac{x_3^2}{A_3^2} = 1$, *and how this number relates to the volume bounded by the surface which is given to be* $\frac{4\pi}{3}A_1A_2A_3$.]

4 The equilibrium position of the free surface of a liquid of depth h is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the free surface and

$$\eta = \eta_0 sin(kx - \omega t),$$

where η_0, k and ω are positive constants. You are given that the velocity potential $\phi = \phi(x, z, t)$ satisfies

$$\phi_{xx} + \phi_{zz} = 0.$$

You are also given that (a) $\phi_z \to 0$ as $z \to -h$; (b) $\phi_z = \eta_t$ at $z = 0$; (c) $\phi_t + g\eta = 0$ at $z = 0$, where g is the constant acceleration of gravity.

(i) Give a brief physical interpretation of (a), (b) and (c).

(3 marks)

(ii) Find $\phi(x, z, t)$ and show that

$$\omega^2 = gk \tanh(kh).$$

(11 marks)

(iii) Consider a particle whose equilibrium position is (x_0, z_0). Suppose its position at time t is $(x_0 + X(t), z_0 + Z(t))$, where the time means of X and Z will be chosen to be zero. Using the velocity potential $\phi(x, z, t)$ and dispersion relation derived in (ii) above, show that

$$\frac{X^2}{a^2} + \frac{Z^2}{b^2} = 1$$

and determine a and b. Explain briefly the result.

(11 marks)

5 (i) Using the method of characteristics integrate the *associated equations* to derive the solution of

$$x^3 z_x = z_y$$

with $z = 1/(1 + x^2)$ on $y = 0$, $-\infty < x < \infty$. Explain why the solution is not defined when $y \geq 1/(2x^2)$.

(8 marks)

 Question 5 continued on next page

5 (continued)

(ii) Using the method of characteristics solve

$$\rho_t + \rho\rho_x = 0$$

with $\rho = f(x)$ at $t = 0$ given in two cases:

(a) $f(x) = 0 \quad (x < 0), \quad f(x) = x \quad (0 \le x < 1), \quad f(x) = 1 \quad (x \ge 1);$

(b) $f(x) = 0 \quad (x < 0), \quad f(x) = -x \quad (0 \le x < 1), \quad f(x) = -1 \quad (x \ge 1).$

(9 marks)

(iii) Determine, whether the solutions found in (ii) break down for any time t, and if so, at what value.

(2 marks)

(iv) Sketch the function $\rho(x)$ at $t = 0$ and the characteristics for (a) and (b), respectively, on $-3 \le x \le 3$.

(6 marks)

End of Question Paper