The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2015–16

WAVES — 2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics.*

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1 (i) *Verify* that the d'Alembert general solution

$$u(x, t) = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions and c is a constant, satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Hence find the transverse displacement, for time $t > 0$, of an infinite stretched string that, at $t = 0$, is at rest with displacement $\sin(x)$, where x is the distance along the string.

(10 marks)

(ii) The transverse displacement u of a stretched string, held fixed at its end-points $x = 0$ and $x = L$, satisfies the wave equation given by (1). At $t = 0$ the displacement is zero. *Verify* that

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin(\alpha_n x) \sin(\alpha_n c t)$$

satisfies equation (1) and the initial and boundary conditions, where B_n and α_n $(n = 1, 2, 3, \ldots)$ are constants to be determined.

If it is further given that, at $t = 0$, the velocity $\partial u/\partial t = f(x)$, find an integral formula for B_n.

(15 marks)

2 A string of mass per unit length ρ is under a tension ρc^2, where ρ and c are constants. Its equilibrium position is $0 \le x \le l$, $y = 0$. The string undergoes transverse vibrations and its displacement is $y(x, t)$, where

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2},$$

and $y(0, t) = y(l, t) = 0$.

(i) *Verify* that all these conditions are satisfied by

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left\{ a_n \cos\left(\frac{n\pi c t}{l}\right) + b_n \sin\left(\frac{n\pi c t}{l}\right) \right\},$$

where $\{a_n\}$, $\{b_n\}$ are constants. [*Note that you are asked to verify this result, not derive it.*] ***(7 marks)***

 Question 2 continued on next page

2 (continued)

(ii) Find $\{a_n\}$ and $\{b_n\}$ for the case when

$$y(x,0) = 0, \quad \dot{y}(x,0) = (V/l^3)x(\tfrac{1}{2}l - x)(l - x).$$

(13 marks)

(iii) Show that the initial kinetic energy of the string is

$$\frac{\rho V^2 l}{1680}.$$

(5 marks)

3 (*A model of a stethoscope.*) Sound waves propagate in the positive Oz direction inside the circular cylinder $r = a$ (where $r^2 = x^2 + y^2$ in standard notation). The velocity potential ϕ satisfies

$$c^2 \left\{ \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} \right\} = \frac{\partial^2 \phi}{\partial t^2}, \tag{2}$$

where the constant c is the speed of sound.

(i) State how c depends on pressure (p) and density (ρ). Determine the value of c for the case when this is

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma},$$

where $\gamma = 1.4$, and p_0 and ρ_0 are the ambient pressure and density with $p_0 \approx 1.013 \times 10^5$ N m^{-2}, $\rho_0 \approx 1.293$ kg m^{-3}.

(6 marks)

(ii) Seek solutions of (2) of the form

$$\phi = g(r)\exp\{i(kz - \omega t)\},$$

where k and ω are real positive constants. Show that

$$g''(r) + \frac{1}{r}g'(r) + m^2 g(r) = 0 \tag{3}$$

where m^2 is a constant, depending on ω, c and k. (You may assume that $m^2 > 0$.)

(7 marks)

 Question 3 continued on next page

3 (continued)

(iii) It is given that ϕ is bounded at $r = 0$, that $\dfrac{\partial \phi}{\partial r} = 0$ at $r = a$, and that the only solution of (3) that is bounded at $r = 0$ must be a multiple of $J_0(mr)$, where $J_0(\xi)$ is the Bessel function of order zero. Show that $m = m_n$ $(n = 1, 2, \ldots)$, where $m_n = \beta_n / a$ and β_n is the nth non-zero root of $J_0'(\xi) = 0$. Given that the β_n are discrete, that $\beta_1 < \beta_2 < \ldots$, and that $\beta_n \to \infty$ as $n \to \infty$, deduce that, for fixed ω, there are a finite number of positive values of k.

(12 marks)

4 The equilibrium position of the free surface of a liquid of infinite depth is $z = 0$, where z is measured vertically upwards. A surface wave causes the displacement of this surface to be $\eta(x, t)$, where x is measured along the undisturbed surface and

$$\eta = a \sin(kx - \omega t),$$

with a, k and ω being positive constants with a small.
The velocity potential is $\phi(x, z, t)$ and satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

You are given that: (a) $\dfrac{\partial \phi}{\partial z} \to 0$ as $z \to -\infty$; (b) $\dfrac{\partial \phi}{\partial z} = \dfrac{\partial \eta}{\partial t}$ at $z = 0$;
(c) $\dfrac{\partial \phi}{\partial t} + g\eta = 0$ at $z = 0$.

(i) Explain briefly the physical meaning of each of (a), (b), and (c).

(5 marks)

(ii) Find $\phi(x, z, t)$ and show that the dispersion relation is

$$\omega^2 = gk.$$

(14 marks)

(iii) Determine the phase velocity c and the group velocity c_g in terms of k. State two quantities that are propagated with speed c_g.

(6 marks)

5 In a model of traffic flow in the direction of Ox, the density of traffic at time t is $\rho(x, t)$, and it is assumed that the velocity of traffic of density ρ is $v = v(\rho)$.

(i) Show that

$$\rho_t + c(\rho)\rho_x = 0,$$

where $c(\rho) = d(\rho v)/d\rho$.

(4 marks)

(ii) Given that $\rho = f(x)$ at $t = 0$ for $-\infty < x < \infty$, and that

$$c(f(\xi)) = F(\xi),$$

show that, for $t \geq 0$, $\rho = f(\xi)$ on the curve $x = \xi + F(\xi)t$.

(7 marks)

(iii) Show that the above solution breaks down on any curve for which $F'(\xi) < 0$.

(2 marks)

(iv) In a particular case

$$v(\rho) = \frac{V}{P}(P - \rho) \qquad (0 \leq \rho \leq P),$$

where V and P are constant. Given that

$$\rho(x, 0) = \begin{cases} 0 & (x \leq 0), \\ \rho_R(x^2/L^2) & (0 \leq x \leq L), \\ \rho_R & (x \geq L), \end{cases}$$

where L and ρ_R are constants with $\rho_R < P$, obtain the solution to the traffic flow equation in (i). Determine when and where the solution first breaks down.

(12 marks)

End of Question Paper