

1. • From (1.11) and (1.5), the general solution of  $\phi_{xx} = \frac{1}{c^2}\phi_{yy}$  is  $\phi = f(x - cy) + g(x + cy)$ .  
 Put  $c = 1$ : the general solution of  $\phi_{xx} = \phi_{yy}$  is therefore

$$\boxed{\phi = f(x - y) + g(x + y)} \tag{*}$$

- To solve the three problems, we use the methods of §1.3 (with necessary changes in notation).

$$(*) \Rightarrow \frac{\partial \phi}{\partial y} = -f'(x - y) + g'(x + y)$$

by the chain rule.

Thus (\*) gives

$$\boxed{\phi(x, 0) = f(x) + g(x); \quad \frac{d\phi}{dy}(x, 0) = -f'(x) + g'(x)} \tag{†}$$

- (i) (†)  $\Rightarrow f(x) + g(x) = 0; -f'(x) + g'(x) = 4x$   
 Integrate the second of these, obtaining  $f(x) - g(x) = A - 2x^2$ , where A is a constant.  
 Thus  $g(x) = f(x) + 2x^2 - A$ .  
 Then  $f(x) + g(x) = 0 \Rightarrow f(x) = -x^2 + \frac{1}{2}A, g(x) = +x^2 - \frac{1}{2}A$ .  
 Thus, from (\*),  $\phi = -(x - y)^2 + (x + y)^2$ , i.e.

$$(i) \quad \boxed{\phi = 4xy}$$

- (ii) (†)  $\Rightarrow f(x) + g(x) = \cos(kx), -f'(x) + g'(x) = k \sin(kx)$   
 Integrate the second of these, obtaining  $f(x) - g(x) = A + \cos(kx)$ , where A is a constant.  
 Thus  $g(x) = f(x) - \cos(kx) - A$ .  
 Then  $f(x) + g(x) = \cos(kx) \Rightarrow f(x) \cos(kx) + \frac{1}{2}A, g(x) = -\frac{1}{2}A$   
 Thus, from (\*),

$$(ii) \quad \boxed{\phi = \cos k(x - y)} \quad \text{(A travelling wave, travelling in positive } x \text{ direction)}$$

- (iii) (†)  $\Rightarrow f(x) + g(x) = 0, -f'(x) + g'(x) = k \sin(kx)$   
 As in (ii),  $f(x) - g(x) = A + \cos(kx), g(x) = f(x) - \cos(kx) - A$   
 Then  $f(x) + g(x) = 0 \Rightarrow f(x) = \frac{1}{2} \cos(kx) + \frac{1}{2}A, g(x) = -\frac{1}{2} \cos(kx) - \frac{1}{2}A$   
 Thus, from (\*),  $\phi = \frac{1}{2}[\cos\{k(x - y)\} - \cos\{k(x + y)\}]$ , i.e.  
 (since  $\cos A - \cos B = 2 \sin \left(\frac{B-A}{2}\right) \sin \left(\frac{A+B}{2}\right)$ )

$$(iii) \quad \boxed{\phi = \sin(kx) \cdot \sin(ky)} \quad \text{(a standing wave)}$$

2. • This PDE is precisely (1.5) whose GS is (1.11), i.e.

$$y = f(x - ct) + g(x + ct) \tag{*}$$

- From (\*)

$$\frac{\partial y}{\partial t} = -cf'(x - ct) + cg'(x + ct).$$

Thus we need  $f(x)$  and  $g(x)$  to satisfy

$$f(x) + g(x) = \begin{cases} 0 & (-\infty < x < a) \\ (a^2 - x^2) & (-a \leq x \leq a) \\ 0 & (a < x < \infty) \end{cases} \quad f'(x) = g'(x)$$

- The second gives  $f(x) = g(x) + A$ , so  $2f(x) - A = y(x, 0)$ , i.e.

$$f(x) = \frac{1}{2}A + \begin{cases} 0 & (-\infty < x < -a) \\ \frac{1}{2}(a^2 - x^2) & (-a \leq x \leq a) \\ 0 & (a < x < \infty) \end{cases} \quad g(x) = f(x) - A$$

Thus

$$f(x - ct) = \frac{1}{2}A + \begin{cases} 0 & (-\infty < x < ct - a) \\ \frac{1}{2}[a^2 - (x - ct)^2] & (ct - a \leq x \leq ct + a) \\ 0 & (ct + a < x < \infty) \end{cases}$$

$$g(x + ct) = -\frac{1}{2}A + \begin{cases} 0 & (-\infty < x < ct - a) \\ \frac{1}{2}[a^2 - (x - ct)^2] & (-ct - a \leq x \leq -ct + a) \\ 0 & (-ct + a < x < \infty) \end{cases}$$

and  $y = f(x - ct) + g(x + ct)$

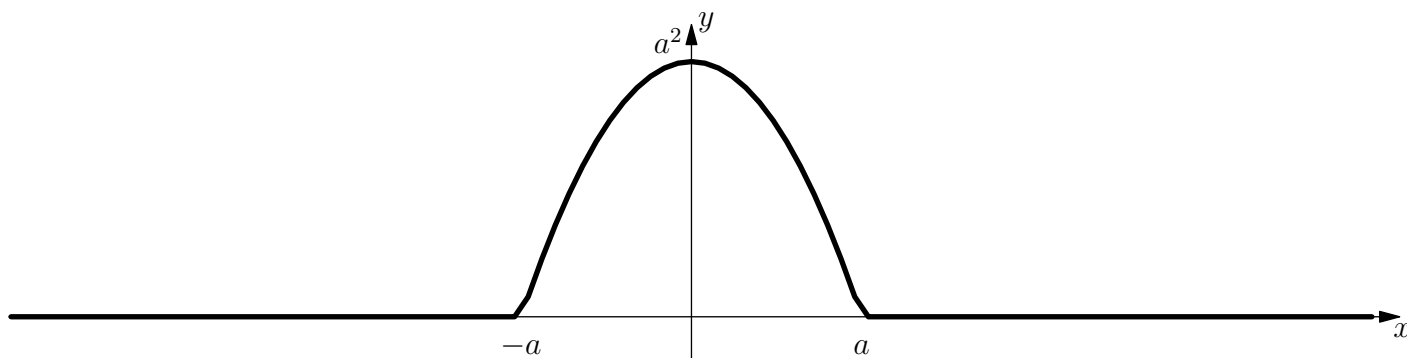
$$\boxed{ct = 0}$$

$$\boxed{ct = \frac{1}{2}a} \left\{ \begin{array}{l} f(x - ct) = \begin{cases} 0 & (-\infty < x < -\frac{1}{2}a) \\ \frac{1}{2}[a^2 - (x - \frac{1}{2}a)^2] & (-\frac{1}{2}a \leq x \leq \frac{3}{2}a) \\ 0 & (\frac{3}{2}a < x < \infty) \end{cases} \\ g(x + ct) = \begin{cases} 0 & (-\infty < x < -\frac{3}{2}a) \\ \frac{1}{2}[a^2 - (x + \frac{1}{2}a)^2] & (-\frac{3}{2}a \leq x \leq \frac{1}{2}a) \\ 0 & (\frac{1}{2}a < x < \infty) \end{cases} \end{array} \right.$$

$$\boxed{ct = 2a} \left\{ \begin{array}{l} f(x - ct) = \begin{cases} 0 & (-\infty < x < a) \\ \frac{1}{2}[a^2 - (x - 2a)^2] & (a \leq x \leq 3a) \\ 0 & (3a < x < \infty) \end{cases} \\ g(x + ct) = \begin{cases} 0 & (-\infty < x < -3a) \\ \frac{1}{2}[a^2 - (x + 2a)^2] & (-3a \leq x \leq -a) \\ 0 & (-a < x < \infty) \end{cases} \end{array} \right.$$

We have ignored A which cancels. Hence the sketch graphs which follow:

$ct = 0$

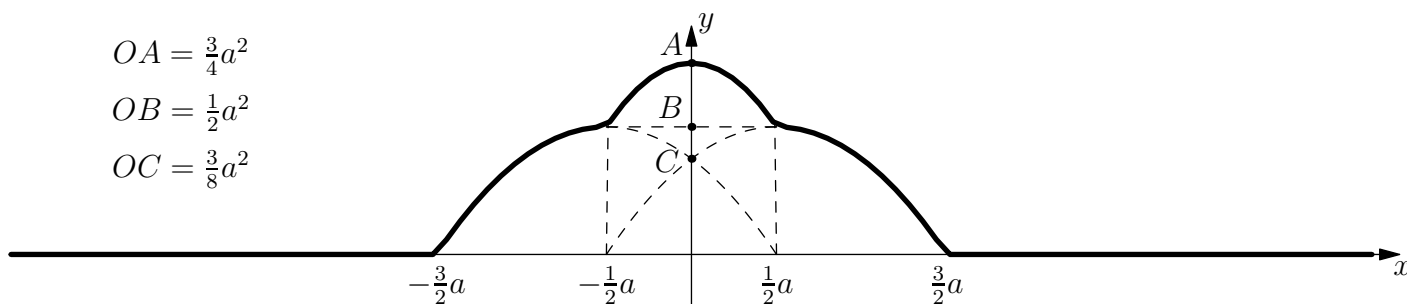


$ct = \frac{1}{2}a$

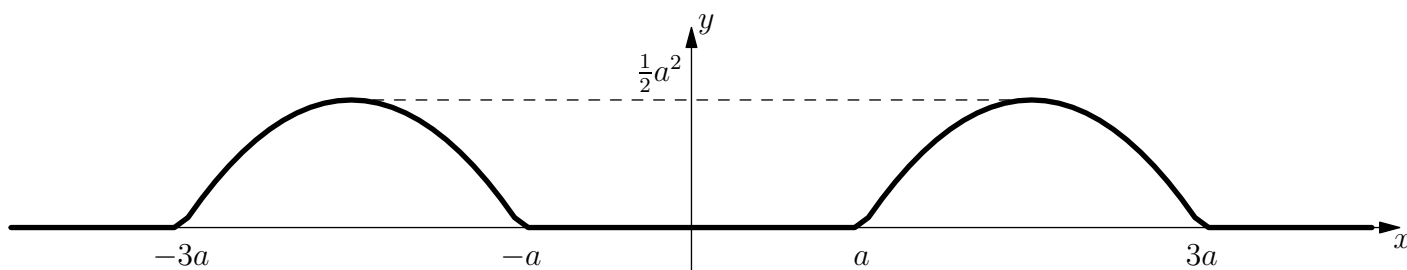
$OA = \frac{3}{4}a^2$

$OB = \frac{1}{2}a^2$

$OC = \frac{3}{8}a^2$



$ct = 2a$



3. (i)

$$u_t = \frac{\partial u}{\partial t} = Akc \sin\{k(x - ct)\}$$

$$u_x = \frac{\partial u}{\partial x} = -Ak \sin\{k(x - ct)\}$$

$$\therefore u_t + au_x = Ak \sin\{k(x - ct)\} \times (c - a)$$

This is zero for all  $x, t \Leftrightarrow \boxed{c = a}$  ( $A$  and  $k$  are arbitrary)

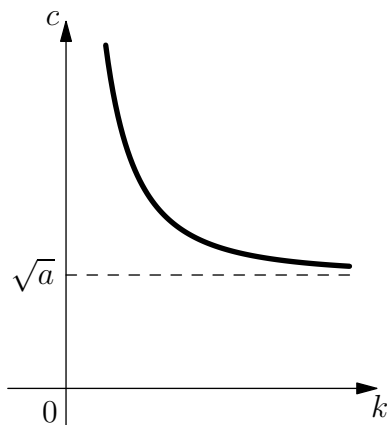
(ii)

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = -Ak^2 c^2 \cos\{k(x - ct)\}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = -Ak^2 \cos\{k(x - ct)\}$$

$$\therefore u_{tt} - au_{xx} + bu = A \cos\{k(x - ct)\} \times [-k^2 c^2 + k^2 a + b]$$

This is zero for all  $x, t \Leftrightarrow \boxed{k \neq 0 \text{ and } c^2 = a + bk^{-2}}$



NB The value of  $A$  is irrelevant because both PDEs are linear.

4. • From (1.7),

$$T = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{1}{2}\rho c^2 \int_{-\infty}^{\infty} \{f'(x - ct)\}^2 dx$$

From (1.8),

$$V = \frac{1}{2}F \int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}F \int_{-\infty}^{\infty} \{f'(x - ct)\}^2 dx$$

These are equal because  $F = \rho c^2$  (and both integrals exist.)

- Likewise,  $y = g(x + ct) \Rightarrow$

$$T = \frac{1}{2}\rho c^2 \int_{-\infty}^{\infty} \{g'(x + ct)\}^2 dx$$

and

$$V = \frac{1}{2}F \int_{-\infty}^{\infty} \{g'(x + ct)\}^2 dx$$

and these are equal because  $F = \rho c^2$

- $y = f(x - ct) + g(x + ct) \Rightarrow$

$$\frac{\partial y}{\partial t} = -c\{f'(x - ct) - g'(x + ct)\}$$

and

$$\frac{\partial y}{\partial x} = \{f'(x - ct) + g'(x + ct)\}.$$

Hence

$$T = \frac{1}{2}\rho c^2 \left[ \int_{-\infty}^{\infty} \{f'(x - ct)\}^2 - 2f'(x - ct)g'(x + ct) + \{g'(x + ct)\}^2 \right] dx$$

$$V = \frac{1}{2}F \left[ \int_{-\infty}^{\infty} \{f'(x - ct)\}^2 + 2f'(x - ct)g'(x + ct) + \{g'(x + ct)\}^2 \right] dx$$

$$F = \rho c^2 \Rightarrow T - V = -2\rho c^2 \int_{-\infty}^{\infty} \{f'(x - ct)g'(x + ct)\} dx \quad \boxed{\neq 0 \text{ in general}}$$

5. • (i)  $r^2 = x^2 + y^2 + z^2 \Rightarrow \frac{\partial}{\partial x}(r^2) = 2x \Rightarrow \frac{\partial}{\partial r}(r^2) \cdot \frac{\partial r}{\partial x} = 2x$  by chain rule  
 $\Rightarrow 2r \frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = xr^{-1}}$

(ii) Thus  $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \frac{\partial r}{\partial x}$  (chain rule) =  $\boxed{xr^{-1} \frac{\partial \phi}{\partial r}}$

(iii)

$$\begin{aligned} \therefore \frac{\partial^2 \phi}{\partial x^2} &= \frac{\partial}{\partial x} \left( xr^{-1} \frac{\partial \phi}{\partial r} \right) = r^{-1} \frac{\partial \phi}{\partial r} + x \frac{\partial}{\partial r} \left( r^{-1} \frac{\partial \phi}{\partial r} \right) \cdot \frac{\partial r}{\partial x} \\ &= r^{-1} \frac{\partial \phi}{\partial r} + x^2 r^{-1} \left( r^{-1} \frac{\partial^2 \phi}{\partial r^2} - r^{-2} \frac{\partial \phi}{\partial r} \right) \\ &= \boxed{\frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{x^2}{r^2} \left( \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} \right)} \end{aligned}$$

• Hence

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{3}{r} \frac{\partial \phi}{\partial r} + \frac{(x^2 + y^2 + z^2)}{r^2} \left( \frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} \right) = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r}.$$

Thus PDE satisfied by  $\phi = \phi(r, t)$  is

$$\boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}} \tag{*}$$

•  $\psi = r\phi \Rightarrow \phi = r^{-1}\psi \Rightarrow \phi_r = r^{-1}\psi_r - r^{-2}\psi$  and  $\phi_{rr} = r^{-1}\psi_{rr} - 2r^{-2}\psi_r + 2r^{-3}\psi$ . Substitute into (\*) to get

$$\begin{aligned} r^{-1}\psi_{rr} - \cancel{2r^{-2}\psi_r} + \cancel{2r^{-3}\psi} + \cancel{2r^{-2}\psi_r} - \cancel{2r^{-3}\psi} &= \frac{1}{c^2} r^{-1} \psi_{tt} \\ \Rightarrow \boxed{\psi_{rr} = \frac{1}{c^2} \psi_{tt}} \end{aligned}$$

• This is structurally identical to (1.5) so its general solution is given by (1.11) with the obvious changes in notion. Thus  $\psi = r\phi = f(r - ct) + g(r + ct) \Rightarrow$  GS of spherical wave equation is

$$\boxed{\phi = \frac{1}{r} \{f(r - ct) + g(r + ct)\}}$$

6. • Substitute  $y = X(x)T(t)$  into the PDE to get

$$X''T = \frac{1}{c^2}X\ddot{T} + \frac{1}{c^2\tau}X\dot{T}$$

- Divide by  $XT$  to get

$$\frac{X''}{X} = \frac{1}{c^2} \left( \frac{\ddot{T} + \tau^{-1}\dot{T}}{T} \right) \quad (*)$$

- The LHS of (\*) depends only on  $x$ , and the RHS of (\*) depends only on  $t$ . Thus (\*) can be true for all  $x$  and  $t$  only if each side of (\*) is a constant.

There are three cases to consider

I: constant  $> 0$ ,  $= p^2$  (say). Thus  $X'' = p^2X \Rightarrow X = A \cosh(px) + B \sinh(px)$ .  
 $y(0) = 0 \Rightarrow A = 0 \Rightarrow X = B \sinh(px)$ .  $y(l) = 0 \Rightarrow B \sinh(pl) = 0 \Rightarrow B = 0$  or  
 $p = 0 \Rightarrow X = 0$  for all  $x$ . **REJECT**

II: constant  $= 0$ . Thus  $X'' = 0 \Rightarrow X = Ax + B$ .  $y(0) = 0 \Rightarrow B = 0 \Rightarrow X = Ax$ .  
 $y(l) = 0 \Rightarrow Al = 0 \Rightarrow A = 0 \Rightarrow X = 0$  for all  $x$ . **REJECT**

III: constant  $< 0$ ,  $= -p^2$  (say). Thus  $X'' = -p^2X \Rightarrow X = A \cos(px) + B \sin(px)$ .  $y(0) = 0 \Rightarrow A = 0 \Rightarrow X = B \sin(px)$ .  $y(l) = 0 \Rightarrow B \sin(pl) = 0 \Rightarrow B = 0$  or  $pl = n\pi$ . Reject  $B = 0$  because that would give  $X = 0$  for all  $x \Rightarrow$

$$X = B \sin\left(\frac{n\pi x}{l}\right) \quad (n = 1, 2, 3, \dots)$$

- With each side of (\*) equal to  $-p^2 = -\frac{n^2\pi^2}{l^2}$ , the equation for  $T$  is

$$\ddot{T} + \frac{1}{\tau}\dot{T} + \left(\frac{n\pi c}{l}\right)^2 T = 0.$$

This is a linear ODE for  $T$  so try  $T \propto e^{mt}$ . This is a solution if

$$m^2 + \frac{1}{\tau}m + \left(\frac{n\pi c}{l}\right)^2 = 0$$

$$\therefore m = \frac{1}{2} \left[ -\frac{1}{\tau} \pm \sqrt{\left\{ \frac{1}{\tau^2} - \left(\frac{2n\pi c}{l}\right)^2 \right\}} \right]$$

- Again there are three cases to consider

(i)

$$2\pi c\tau > l \Rightarrow \frac{1}{\tau^2} - \left(\frac{2n\pi c}{l}\right)^2 < 0 \text{ for all } n. \text{ Let } \frac{1}{\tau^2} - \left(\frac{2n\pi c}{l}\right)^2 = -\frac{4\lambda_n^2}{\tau^2} \Rightarrow$$

$$m = \frac{1}{\tau}[-\frac{1}{2} \pm i\lambda_n] \Rightarrow T = e^{-\frac{t}{2\tau}} \left\{ \alpha_n \cos\left(\lambda_n \frac{t}{\tau}\right) + \beta_n \sin\left(\lambda_n \frac{t}{\tau}\right) \right\} \text{ and}$$

$$y(x, t) = e^{-\frac{t}{2\tau}} \sin\left(\frac{n\pi x}{l}\right) \left\{ \alpha_n \cos\left(\lambda_n \frac{t}{\tau}\right) + \beta_n \sin\left(\lambda_n \frac{t}{\tau}\right) \right\}$$

(ii)  $2\pi c\tau < l$  and there is no integer  $n$  with  $2n\pi c\tau = l$ .Thus there will be a finite number of integers  $\{1, 2, \dots, N\}$  for which

$$\frac{1}{\tau} > \left(\frac{2n\pi c}{l}\right)^2 \Rightarrow \text{for each such } n, m = -m_n, -m_n$$

where  $m_n$  and  $M_n$  are real and positive. For other  $n$ , we return to case (i). The new types of solution are

$$y(x, t) = \sin\left(\frac{n\pi x}{l}\right) \left\{ a_n e^{-m_n t} + A_n e^{-M_n t} \right\} \quad (n = 1, 2, \dots, N)$$

(iii)  $2\pi c\tau \leq l$  and there is an integer  $N$  with  $2\pi N c\tau = l$ . For  $n < N$  (possibly an empty set) we get solutions as in (ii), and for  $n > N$ , as in (i). For  $n = N$ , we have  $m = -\frac{1}{2\tau}$  as repeated root. Thus for this  $n = N$

$$y(x, t) = \sin\left(\frac{N\pi x}{l}\right) e^{-\frac{t}{2\tau}} (At + B)$$

[In cases (ii) and (iii) the friction is relatively so large that some modes decay to zero as  $t$  increases without oscillation.]