1. • From (1.11) and (1.5), the general solution of $\phi_{xx} = \frac{1}{c^2}\phi_{yy}$ is $\phi = f(x - cy) + g(x + cy)$.
Put $c = 1$: the general solution of $1\phi_{xx} = \phi_{yy}$ is therefore

$$\boxed{\phi = f(x - y) + g(x + y)} \qquad (*)$$

• To solve the three problems, we use the methods of §1.3 (with necessary changes in notation).

$$(*) \Rightarrow \frac{\partial \phi}{\partial y} = -f'(x - y) + g'(x + y)$$

by the chain rule.
Thus $(*)$ gives

$$\boxed{\phi(x, 0) = f(x) + g(x); \quad \frac{d\phi}{dy}(x, 0) = -f'(x) + g(x)} \qquad (\ddagger)$$

• (i) $(\ddagger) \Rightarrow f(x) + g(x) = 0;\ -f'(x) + g'(x) = 4x$
Integrate the second of these, obtaining $f(x) - g(x) = A - 2x^2$, where A is a constant.
Thus $g(x) = f(x) + 2x^2 - A$.
Then $f(x) + g(x) = 0 \Rightarrow f(x) = -x^2 + \frac{1}{2}A,\ g(x) = +x^2 - \frac{1}{2}A$.
Thus, from $(*)$, $\phi = -(x - y)^2 + (x + y)^2$, i.e.

$$(i)\ \boxed{\phi = 4xy}$$

• (ii) $(\ddagger) \Rightarrow f(x) + g(x) = \cos(kx),\ -f('x) + g'(x) = k\sin(kx)$
Integrate the second of these, obtaining $f(x) - g(x) = A + \cos(kx)$, where A is a constant.
Thus $g(x) = f(x) - \cos(kx) - A$.
Then $f(x) + g(x) = \cos(kx) \Rightarrow f(x)\cos(kx) + \frac{1}{2}A,\ g(x) = -\frac{1}{2}A$
Thus, from $(*)$,

$$(ii)\ \boxed{\phi = \cos k(x - y)}$$

(A travelling wave, travelling in positive x direction)

• (iii) $(\ddagger) \Rightarrow f(x) + g(x) = 0,\ -f('x) + g'(x) = k\sin(kx)$
As in (ii), $f(x) - g(x) = A + \cos(kx)$, $g(x) = f(x) - \cos(kx) - A$
Then $f(x) + g(x) = 0 \Rightarrow f(x) = \frac{1}{2}\cos(kx) + \frac{1}{2}A,\ g(x) = -\frac{1}{2}\cos(kx) - \frac{1}{2}A$
Thus, from $(*)$, $\phi = \frac{1}{2}[\cos\{k(x - y)\} - \cos\{k(x + y)\}]$, i.e.
(since $\cos A - \cos B = 2\sin\left(\frac{B-A}{2}\right)\sin\left(\frac{A+B}{2}\right)$)

$$(iii)\ \boxed{\phi = \sin(kx) \cdot \sin(ky)}$$

(a standing wave)

2. • This PDE is precisely (1.5) whose GS is (1.11), i.e.

$$y = f(x-ct) + g(x+ct) \tag{$*$}$$

• From ($*$)

$$\frac{\partial y}{\partial t} = -cf'(x-ct) + cg'(x+ct).$$

Thus we need $f(x)$ and $g(x)$ to satisfy

$$f(x)+g(x) = \begin{cases} 0 & (-\infty < x < a) \\ (a^2-x^2) & (-a \le x \le a) \\ 0 & (a < x < \infty) \end{cases} \qquad f'(x) = g'(x)$$

• The second gives $f(x) = g(x) + A$, so $2f(x) - A = y(x,0)$, i.e.

$$f(x) = \frac{1}{2}A + \begin{cases} 0 & (-\infty < x < -a) \\ \frac{1}{2}(a^2-x^2) & (-a \le x \le a) \\ 0 & (a < x < \infty) \end{cases} \qquad g(x) = f(x) - A$$

Thus

$$\boxed{\begin{array}{rcll} f(x-ct) & = & \frac{1}{2}A + & \begin{cases} 0 & (-\infty < x < ct-a) \\ \frac{1}{2}[a^2-(x-ct)^2] & (ct-a \le x \le ct+a) \\ 0 & (ct+a < x < \infty) \end{cases} \\ \\ g(x+ct) & = & -\frac{1}{2}A + & \begin{cases} 0 & (-\infty < x < -ct-a) \\ \frac{1}{2}[a^2-(x+ct)^2] & (-ct-a \le x \le -ct+a) \\ 0 & (-ct+a < x < \infty) \end{cases} \\ \\ & & \text{and} & y = f(x-ct)+g(x+ct) \end{array}}$$

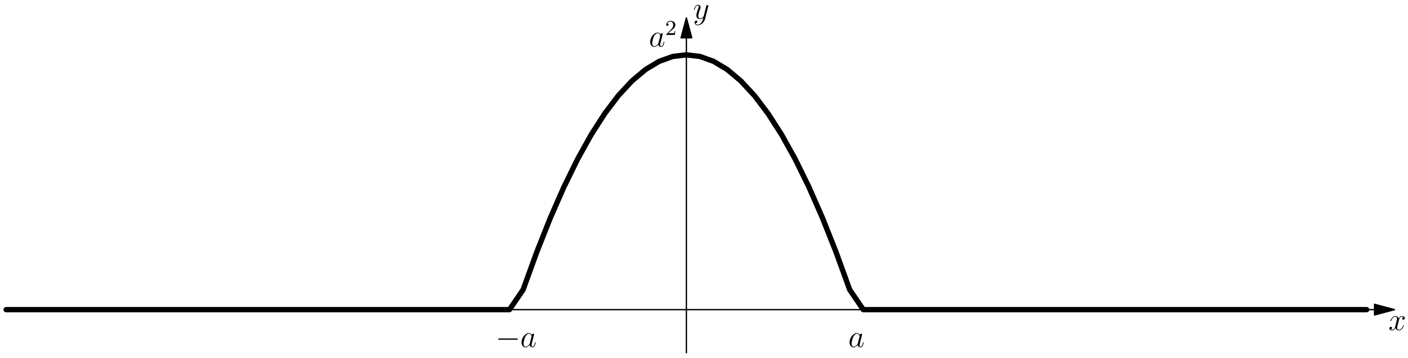
$\boxed{ct = 0}$

$$\boxed{ct = \tfrac{1}{2}a} \quad \begin{cases} f(x-ct) = \begin{cases} 0 & (-\infty < x < -\frac{1}{2}a) \\ \frac{1}{2}\left[a^2 - \left(x-\frac{1}{2}a\right)^2\right] & (-\frac{1}{2}a \le x \le \frac{3}{2}a) \\ 0 & \left(\frac{3}{2}a < x < \infty\right) \end{cases} \\ \\ g(x+ct) = \begin{cases} 0 & (-\infty < x < -\frac{3}{2}a) \\ \frac{1}{2}\left[a^2 - \left(x+\frac{1}{2}a\right)^2\right] & (-\frac{3}{2}a \le x \le \frac{1}{2}a) \\ 0 & \left(\frac{1}{2}a < x < \infty\right) \end{cases} \end{cases}$$

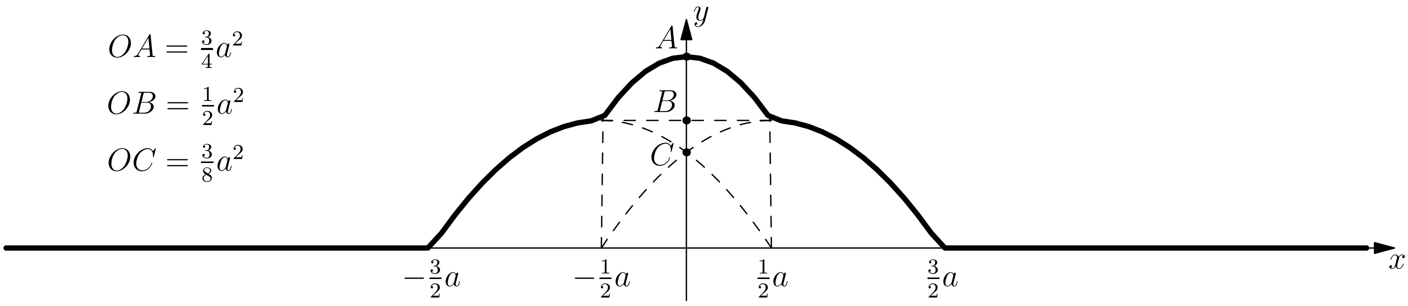
$$\boxed{ct = 2a} \quad \begin{cases} f(x-ct) = \begin{cases} 0 & (-\infty < x < a) \\ \frac{1}{2}\left[a^2 - (x-2a)^2\right] & (a \le x \le 3a) \\ 0 & (3a < x < \infty) \end{cases} \\ \\ g(x+ct) = \begin{cases} 0 & (-\infty < x < -3a) \\ \frac{1}{2}\left[a^2 - (x+2a)^2\right] & (-3a \le x \le -a) \\ 0 & (-a < x < \infty) \end{cases} \end{cases}$$

We have ignored A which cancels. Hence the sketch graphs which follow:

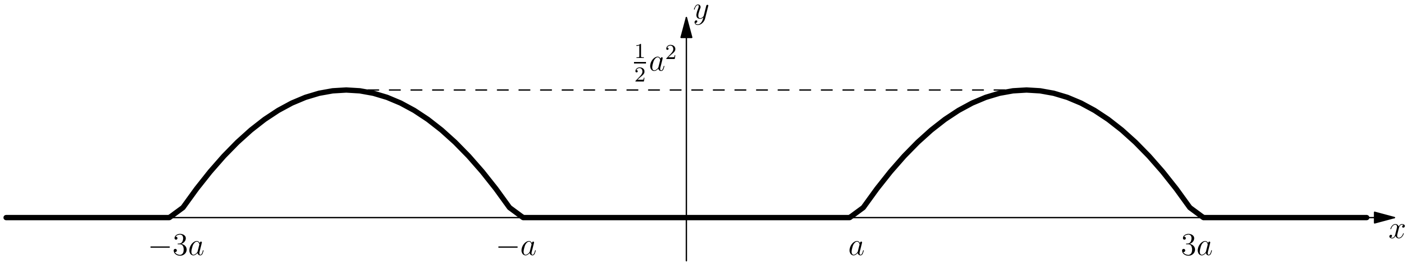
$\boxed{ct = 0}$



$\boxed{ct = \frac{1}{2}a}$



$\boxed{ct = 2a}$



3. (i)

$$
\begin{aligned}
u_t &= \frac{\partial u}{\partial t} = Akc\sin\{k(x-ct)\} \\
u_x &= \frac{\partial u}{\partial x} = -Ak\sin\{k(x-ct)\} \\
\therefore u_t + au_x &= Ak\sin\{k(x-ct)\} \times (c-a)
\end{aligned}
$$

This is zero for all $x, t \Leftrightarrow \boxed{c = a}$ (A and k are arbitrary)

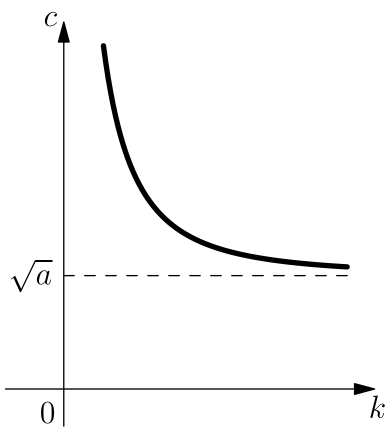
(ii)

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = -Ak^2c^2\cos\{k(x-ct)\}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = -Ak^2\cos\{k(x-ct)\}$$

$$\therefore u_{tt} - au_{xx} + bu = A\cos\{k(x-ct)\} \times \left[-k^2c^2 + k^2a + b\right]$$

This is zero for all $x, t \Leftrightarrow \boxed{k \neq 0 \text{ and } c^2 = a + bk^{-2}}$



NB The value of A is irrelevant because both PDEs are linear.

4. • From (1.7),

$$T = \frac{1}{2}\rho \int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{1}{2}\rho c^2 \int_{-\infty}^{\infty} \{f'(x-ct)\}^2 dx$$

From (1.8),

$$V = \frac{1}{2}F \int_{-\infty}^{\infty} \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}F \int_{-\infty}^{\infty} \{f'(x-ct)\}^2 dx$$

These are $\boxed{\text{equal}}$ because $F = \rho c^2$ (and both integrals exist.)

• Likewise, $y = g(x+ct) \Rightarrow$

$$T = \frac{1}{2}\rho c^2 \int_{-\infty}^{\infty} \{g'(x+ct)\}^2 dx$$

and

$$V = \frac{1}{2}F \int_{-\infty}^{\infty} \{g'(x+ct)\}^2 dx$$

and these are $\boxed{\text{equal}}$ because $F = \rho c^2$

• $y = f(x-ct) + g(x+ct) \Rightarrow$

$$\frac{\partial y}{\partial t} = -c\{f'(x-ct) - g'(x+ct)\}$$

and

$$\frac{\partial y}{\partial x} = \{f'(x-ct) + g'(x+ct)\}.$$

Hence

$$\begin{aligned} T &= \tfrac{1}{2}\rho c^2 \left[\int_{-\infty}^{\infty} [\{f'(x-ct)\}^2 - 2f'(x-ct)g'(x+ct) + \{g'(x+ct)\}^2\right] dx \\ V &= \tfrac{1}{2}F \left[\int_{-\infty}^{\infty} [\{f'(x-ct)\}^2 + 2f'(x-ct)g'(x+ct) + \{g'(x+ct)\}^2\right] dx \end{aligned}$$

$$F = \rho c^2 \Rightarrow T - V = -2\rho c^2 \int_{-\infty}^{\infty} \{f'(x-ct)g'(x+ct)\} dx \ \boxed{\neq 0 \text{ in general}}$$

5. • (i) $r^2 = x^2 + y^2 + z^2 \Rightarrow \dfrac{\partial}{\partial x}(r^2) = 2x \Rightarrow \dfrac{\partial}{\partial r}(r^2) \cdot \dfrac{\partial r}{\partial x} = 2x$ by chain rule

$$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = xr^{-1}}$$

(ii) Thus $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial r}\dfrac{\partial r}{\partial x}$ (chain rule) $= \boxed{xr^{-1}\dfrac{\partial \phi}{\partial r}}$

(iii)

$$\begin{aligned}\therefore \quad \frac{\partial^2 \phi}{\partial x^2} &= \frac{\partial}{\partial x}\left(xr^{-1}\frac{\partial \phi}{\partial r}\right) = r^{-1}\frac{\partial \phi}{\partial r} + x\frac{\partial}{\partial r}\left(r^{-1}\frac{\partial \phi}{\partial r}\right)\cdot\frac{\partial r}{\partial x} \\ &= r^{-1}\frac{\partial \phi}{\partial r} + x^2r^{-1}\left(r^{-}1\frac{\partial^2 \phi}{\partial r^2} - r^{-2}\frac{\partial \phi}{\partial r}\right) \\ &= \boxed{\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{x^2}{r^2}\left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right)}\end{aligned}$$

• Hence

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{3}{r}\frac{\partial \phi}{\partial r} + \frac{(x^2+y^2+z^2)}{r^2}\left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right) = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r}.$$

Thus PDE satisfied by $\phi = \phi(r, t)$ is

$$\boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}} \qquad (*)$$

• $\psi = r\phi \Rightarrow \phi = r^{-1}\psi \Rightarrow \phi r = r^{-1}\psi_r - r^{-2}\psi$ and $\phi_{rr} = r^{-1}\psi_{rr} - 2r^{-2}\psi_r + 2r^{-3}\psi$. Substitute into $(*)$ to get

$$r^{-1}\psi_{rr} - \cancel{2r^{-2}\psi_r} + \cancel{2r^{-3}\psi} + \cancel{2r^{-2}\psi_r} - \cancel{2r^{-3}\psi} = \frac{1}{c^2}r^{-1}\psi_{tt}$$

$$\Rightarrow \boxed{\psi_{rr} = \frac{1}{c^2}\psi_{tt}}$$

• This is structurally identical to (1.5) so its general solution is given by (1.11) with the obvious changes in notion. Thus $\psi = r\phi = f(r-ct) + g(r+ct) \Rightarrow$ GS os spherical wave equation is

$$\boxed{\phi = \frac{1}{r}\{f(r-ct) + g(r+ct)\}}$$

6. • Substitute $y = X(x)T(t)$ into the PDE to get

$$X''T = \frac{1}{c^2}X\ddot{T} + \frac{1}{c^2\tau}X\dot{T}$$

• Divide by XT to get

$$\frac{X''}{X} = \frac{1}{c^2}\left(\frac{\ddot{T} + \tau^{-1}\dot{T}}{T}\right) \qquad (*)$$

• The LHS of$(*)$ depends only on x, and the RHS of $(*)$ depends only on t. Thus $(*)$ can be true for all x and t only if each side of $(*)$ is a constant.
There are three cases to consider

I: constant > 0, $= p^2$ (say). Thus $X'' = p^2X \Rightarrow X = A\cosh(ph) + B\sinh(px)$. $y(0) = 0 \Rightarrow A = 0 \Rightarrow X = B\sinh(px)$. $y(l) = 0 \Rightarrow B\sinh(pl) = 0 \Rightarrow B = 0$ or $p = 0 \Rightarrow X = 0$ for all x. $\boxed{\text{REJECT}}$

II: constant $= 0$. Thus $X'' = 0 \Rightarrow X = Ax + B$. $y(0) = 0 \Rightarrow B = 0 \Rightarrow X = Ax$. $y(l) = 0 \Rightarrow Al = 0 \Rightarrow A = 0 \Rightarrow X = 0$ for all x. $\boxed{\text{REJECT}}$

III: constant< 0, $= -p^2$ (say). Thus $X'' = -p^2X \Rightarrow X = A\cos(px) + B\sin(px)$. $y(0) = 0 \Rightarrow A = 0 \Rightarrow X = B\sin(px)$. $y(l) = 0 \Rightarrow B\sin(pl) = 0 \Rightarrow B = 0$ or $pl = n\pi$. Reject $B = 0$ because that would give $X = 0$ for all $x \Rightarrow$

$$X = B\sin\left(\frac{n\pi x}{l}\right) \quad (n = 1, 2, 3, ...)$$

• With each side of $(*)$ equal to $-p^2 = -\frac{n^2\pi^2}{l^2}$, the equation for T is

$$\ddot{T} + \frac{1}{\tau}\dot{T} + \left(\frac{n\pi c}{l}\right)^2 T = 0.$$

This is a linear ODE for T so try $T \propto e^{mt}$. This is a solution if

$$m^2 + \frac{1}{\tau}m + \left(\frac{n\pi c}{l}\right)^2 = 0$$

$$\therefore m = \frac{1}{2}\left[\quad -\frac{1}{\tau} \pm \sqrt{\left\{\frac{1}{\tau^2} - \left(\frac{2n\pi c}{l}\right)^2\right\}}\quad\right]$$

- Again there are three cases to consider

  (i)

$$2\pi c\tau > l \Rightarrow \frac{1}{\tau^2} - \left(\frac{2n\pi c}{l}\right)^2 < 0 \text{ for all } n. \text{ Let } \frac{1}{\tau^2} - \left(\frac{2n\pi c}{l}\right)^2 = -\frac{4\lambda_n^2}{\tau^2} \Rightarrow$$

$$m = \frac{1}{\tau}[-\frac{1}{2} \pm i\lambda_n] \Rightarrow T = e^{-\frac{t}{2\tau}}\{\alpha_n cos\left(\lambda_n \frac{t}{\tau}\right) + \beta_n sin\left(\lambda_n \frac{t}{\tau}\right)\} \text{ and}$$

$$\boxed{y(x,t) = e^{-\frac{t}{2\tau}} \sin\left(\frac{n\pi x}{l}\right)\{\alpha_n \cos\left(\lambda_n \frac{t}{\tau}\right) + \beta_n \sin\left(\lambda_n \frac{t}{\tau}\right)\}}$$

  (ii) $2\pi c\tau < l$ and there is no integer n with $2n\pi c\tau = l$.
  Thus there will be a finite number of integers $\{1, 2, ..., N\}$ for which

$$\frac{1}{\tau} > \left(\frac{2n\pi c}{l}\right)^2 \Rightarrow \text{ for each such } n, \ m = -m_n, \ -m_n$$

  where m_n and M_n are real and positive. For other n, we return to case (i). The new types of solution are

$$\boxed{y(x,t) = sin\left(\frac{n\pi x}{l}\right)\left\{a_n e^{-m_n t} + A_n e^{-M_n t}\right\}} \quad (n = 1, 2, ...N)$$

  (iii) $2\pi c\tau \leq l$ and there is an integer N with $2\pi N c\tau = l$. For $n < N$ (possibly an empty set) we get solutions as in (ii), and for $n > N$, as in (i). For $n = N$, we have $m = -\frac{1}{2\tau}$ as repeated root. Thus for this $n = N$

$$\boxed{y(x,t) = \sin\left(\frac{N\pi x}{l}\right) e^{-\frac{t}{2\tau}}(At+B)}$$

  [In cases (ii) and (iii) the friction is relatively so large that some modes decay to zero as t increases without oscillation.]