1. • Because S_1 (and $1+\frac{1}{3^2}+\frac{1}{5^2}+...$) are absolutely convergent series, the order of the terms can be changed without changing the value of the sum. Thus

$$S_1 = \left(1+\frac{1}{3^2}+\frac{1}{5^2}+...\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+...\right)$$

$$= \left(1+\frac{1}{3^2}+\frac{1}{5^2}+...\right)+\frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+...\right)$$

$$\Longrightarrow \boxed{S_1 = \left(1+\frac{1}{3^2}+\frac{1}{5^2}+...\right)+\frac{1}{4}S_1}$$

Thus

$$\frac{3S_1}{4} = \left(1+\frac{1}{3^2}+\frac{1}{5^2}+...\right) = \frac{\pi^2}{8} \ \{\text{by}(2.23)\}$$

$$\therefore \boxed{S_1 = \frac{\pi^2}{6}}$$

• $$\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2} = \frac{\sin^2\left(\frac{\pi}{3}\right)}{1^2}+\frac{\sin^2\left(\frac{2\pi}{3}\right)}{2^2}+\frac{\sin^2(\pi)}{3^2}+\frac{\sin^2\left(\frac{4\pi}{3}\right)}{4^2}+...$$

$$= \frac{3}{4}\left(1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}+...\right) \quad \text{since} \quad \sin^2\left(\frac{\pi}{3}\right)=\sin^2\left(\frac{2\pi}{3}\right)=\sin^2\left(\frac{4\pi}{3}\right)=...=\frac{3}{4}$$

and $\sin\pi = \sin 2\pi = ... = 0$. Thus $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2} = \frac{3}{4}S_1 - \frac{3}{4}\left(\frac{1}{3^2}+\frac{1}{6^2}+...\right)$

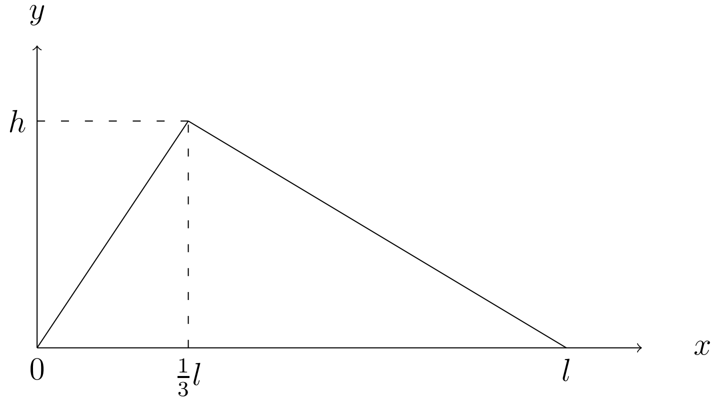
$$= \frac{3}{4}S_1 - \frac{3}{4}\cdot\frac{1}{9}S_1 = \frac{2}{3}S_1 = \boxed{\frac{\pi^2}{9}}$$

2. • Put $t = 0$ in the given series.

Thus $y(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{l}\right)$.

Also, we are given that $y(x,0) = \begin{cases} 3hx/l & (0 \leqslant x \leqslant \frac{1}{3}l) \\ 3h(l-x)/2l & (\frac{1}{3}l \leqslant x \leqslant l) \end{cases}$

and that $\dot{y}(x,0) = 0$.



The second condition is automatically satisfied by the given series. Thus, as in §2.2 [1],

$$\alpha_m = \frac{2}{l}\int_0^l y(x,0)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$= \frac{6h}{l^2}\int_0^{\frac{1}{3}l} x\sin\left(\frac{m\pi x}{l}\right)dx + \frac{3h}{l^2}\int_{\frac{1}{3}l}^{l}(l-x)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$= \frac{6h}{l^2}\left\{\left[-\frac{lx}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\right]_0^{\frac{1}{3}l} + \frac{l}{m\pi}\int_0^{\frac{1}{3}l}\cos\left(\frac{m\pi x}{l}\right)dx\right\}$$

$$+ \frac{3h}{l^2}\left\{\left[\frac{-l(l-x)}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\right]_{\frac{1}{3}l}^{l} - \frac{l}{m\pi}\int_{\frac{1}{3}l}^{l}\cos\left(\frac{m\pi x}{l}\right)dx\right\}$$

$$= \frac{6h}{l^2}\left\{\frac{-l^2}{3m\pi}\cos\left(\frac{m\pi}{3}\right) + \left(\frac{l}{m\pi}\right)^2\sin\left(\frac{m\pi}{3}\right)\right\} + \frac{3h}{l^2}\left\{\frac{2l^2}{3m\pi}\cos\left(\frac{m\pi}{3}\right) + \left(\frac{l}{m\pi}\right)^2\sin\left(\frac{m\pi}{3}\right)\right\}$$

$$\therefore \boxed{\alpha_m = \frac{9h}{m^2\pi^2}\sin\left(\frac{m\pi}{3}\right)}$$

• By (2.20), the energy E_n in the n^{th} normal mode is $\frac{\rho\pi^2c^2n^2\alpha_n^2}{4l}$

$$\therefore \boxed{E_n = \frac{81\rho c^2h^2}{4\pi^2 l}\frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2}}$$

• The total energy is $E = \sum_{n=1}^{\infty} E_n$. By the last result in Q1,

$$E = \frac{81\rho c^2 h^2}{4\pi^2 l} \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2} = \boxed{\frac{9\rho c^2 h^2}{4l}}$$

- This must be equal to the work done in displacing the string to its initial position because energy is conserved. By (1.8) this work done is:

$$V = \frac{1}{2}F\left\{ \int_0^{\frac{1}{3}l} \left(\frac{\partial y}{\partial x}\right)^2 dx + \int_{\frac{1}{3}l}^{l} \left(\frac{\partial y}{\partial x}\right)^2 dx \right\}$$

$$= \frac{1}{2}\rho c^2 \left\{ \int_0^{\frac{1}{3}l} \left(\frac{3h}{l}\right)^2 dx + \int_{\frac{1}{3}l}^{l} \left(\frac{3h}{2l}\right)^2 dx \right\}$$

$$= \frac{1}{2}\rho c^2 \left\{ \frac{9h^2}{l^2}\frac{l}{3} + \frac{9h^2}{4l^2}\frac{2l}{3} \right\} = \boxed{\frac{9\rho c^2 h^2}{4l}} \text{ as required.}$$

[N.B. We have shown that

$$y(x,t) = \left(\frac{9h}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n^2} \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$$

Hence

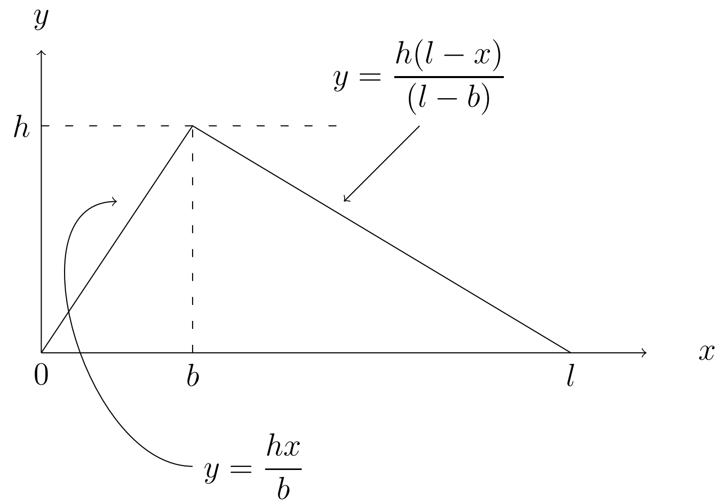
$$y\left(\frac{1}{3}l, 0\right) = \left(\frac{9h}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2} (\bullet)$$

But this is equal to h (given). Hence

$$\left(\frac{9h}{\pi^2}\right) \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2} = h \Longrightarrow \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi}{3}\right)}{n^2} = \frac{\pi^2}{9} \quad \text{which is the last result in Q1]}$$

3. We use the same series as in Q2. Then, as in Q2, we have (see sketch below):

$$\alpha_m = \frac{2h}{lb}\int_0^b x\sin\left(\frac{m\pi x}{l}\right)dx + \frac{2h}{l\,(l-b)}\int_b^l (l-x)\sin\left(\frac{m\pi x}{l}\right)dx$$



$$= \frac{2h}{lb}\left\{\left[-\frac{lx}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\right]_0^b + \frac{l}{\pi m}\int_0^b \cos\left(\frac{m\pi x}{l}\right)dx\right\}$$

$$+\frac{2h}{l\,(l-b)}\left\{\left[\frac{-l\,(l-x)}{m\pi}\cos\left(\frac{m\pi x}{l}\right)\right]_b^l - \frac{l}{m\pi}\int_b^l \cos\left(\frac{m\pi x}{l}\right)dx\right\}$$

$$= \frac{-2h}{m\pi}\cos\left(\frac{m\pi b}{l}\right) + \frac{2hl}{m^2\pi^2 b}\sin\left(\frac{m\pi b}{l}\right) + \frac{2h}{m\pi}\cos\left(\frac{m\pi b}{l}\right) + \frac{2hl}{m^2\pi^2\,(l-b)}\sin\left(\frac{m\pi b}{l}\right)$$

$$= \frac{2hl}{m^2\pi^2}\left[\frac{1}{b} + \frac{1}{l-b}\right]\sin\left(\frac{m\pi b}{l}\right)$$

$$\therefore \boxed{\alpha_m = \frac{2hl^2}{m^2\pi^2 b\,(l-b)}\sin\left(\frac{m\pi b}{l}\right)}$$

$$\therefore y\,(x,t) = \frac{2hl^2}{\pi^2 b\,(l-b)}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{n\pi b}{l}\right)}{n^2}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

Put $t = 0$ and $y = h$, $x = b$ to get

$$h = \frac{2hl^2}{\pi^2 b\,(l-b)}\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi b}{l}\right)}{n^2} \Longrightarrow \boxed{\sum_{n=1}^{\infty}\frac{\sin^2\left(\frac{n\pi b}{l}\right)}{n^2} = \frac{b\,(l-b)\,\pi^2}{2l^2}}$$

4.
- $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$
  Now $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$
  $\therefore \sin 3\theta = 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta = 2\sin\theta\left(1-\sin^2\theta\right) + \sin\theta - 2\sin^3\theta$
  $\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta$
  $\therefore \boxed{\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta}\ (*)$
- We are given that

$$y(x,t) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right)$$

This satisfies $y(0,t) = y(l,t) = 0$ for all t, and $y(x,0) = 0$ for all x.
There remains only to satisfy

$$\dot{y}(x,0) = V\sin^3\left(\frac{\pi x}{l}\right) \text{ i.e. } V\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} \frac{n\pi c\beta_n}{l}\sin\left(\frac{n\pi x}{l}\right)$$

From (*) this means

$$\frac{3V}{4}\sin\left(\frac{\pi x}{l}\right) - \frac{V}{4}\sin\left(\frac{3\pi x}{l}\right) = \sum_{n=1}^{\infty} \frac{n\pi c\beta_n}{l}\sin\left(\frac{n\pi x}{l}\right)$$

By inspection $(*)$, this is satisfied by

$$\frac{\pi c\beta_1}{l} = \frac{3V}{4}, \quad \frac{3\pi c\beta_3}{l} = \frac{-V}{4}, \quad \beta_n = 0 \quad (n \neq 1,3)$$

$$\therefore \boxed{y(x,t) = \frac{Vl}{12\pi c}\left\{9\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi ct}{l}\right)\right\}}$$

$(*)$ The same result is of course obtained much more lengthily using integration in the standard way.

5. Since $y(0,t) = y(l,t) = 0$ for all t, and $y(x,0) = 0$ for all x, we can use the same series as in Q4, and we need to choose the $\{\beta_n\}$ so that

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \beta_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

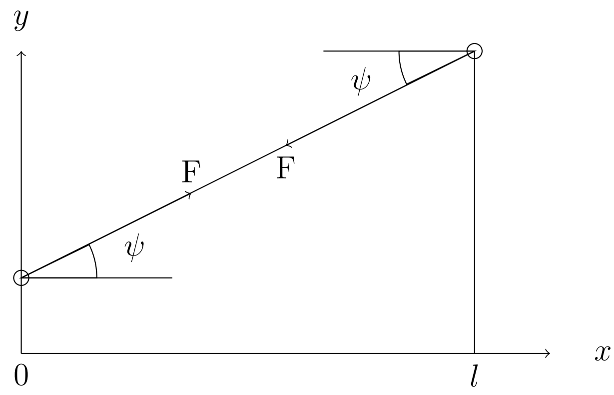
$$\therefore \quad \frac{n\pi c \beta_n}{l} \cdot \frac{l}{2} = \int_0^l \dot{y}(x,0) \sin\left(\frac{n\pi x}{l}\right) dx = V \int_{d-a}^{d+a} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\Longrightarrow \beta_n = \frac{2V}{n\pi c} \cdot \frac{l}{n\pi} \left[\cos\left(\frac{n\pi(d-a)}{l}\right) - \cos\left(\frac{n\pi(d+a)}{l}\right) \right]$$

$$= \frac{4Vl}{c\pi^2} \, \frac{\sin\left(\frac{n\pi d}{l}\right) \sin\left(\frac{n\pi a}{l}\right)}{n^2} \Longrightarrow \boxed{\text{given result}}$$

(This models a string struck by a "hammer" of width $2a$ with speed V centred on the point $x = d$.)

6. • Consider either ring. Since each has zero mass, Newton's Second Law $\Rightarrow$
$x\ddot{y} =$ vertical component of force $= \pm F \sin\psi$
$\therefore \sin\psi = 0$ at $x = 0, l$
But $\psi \ll 1 \Rightarrow \sin\psi \approx \tan\psi = \dfrac{\partial y}{\partial x} \Rightarrow \boxed{\dfrac{\partial y}{\partial x} = 0 \text{ at } x = 0, l}$ $(*)$



Seek separable solutions of $\dfrac{\partial^2 y}{\partial^2 x} = \dfrac{1}{c^2}\dfrac{\partial^2 y}{\partial^2 t}$ of the form $y = X(x)\,T(t)$.
As in §1.4, $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{\ddot{T}}{T}$ and each side must be constant.

There are three cases:

CASE I constant $> 0 = p^2 \therefore X'' = p^2 X \Rightarrow X = A\cosh(px) + B\sinh(px)$
$(*)$ at $x = 0 \Rightarrow B = 0, X = A\cosh(px)$
$(*)$ at $x = l \Rightarrow pA\sinh(pl) = 0 \Rightarrow A = 0\,(\text{since}\, p > 0) \Rightarrow$ no motion REJECT

CASE II constant $= 0 \therefore X'' = 0, \ddot{T} = 0 \Rightarrow T = Ct + D$
This is `NOT a vibration` $\Rightarrow$ REJECT

CASE III constant $< 0 = -p^2 \therefore X'' = -p^2 X\ ,\ \ddot{T} = -p^2 c^2 T$
$\therefore X = A\cos(px) + B\sin(px)\ ,\ T = \alpha\cos(pct) + \beta\sin(pct)$
$(*)$ at $x = 0 \Rightarrow B = 0, X = A\cos(px)$
$(*)$ at $x = l \Rightarrow -pA\sin(pl) = 0$. We get no motion if $A = 0$ (as in CASE I), so $\sin(pl) = 0 \Rightarrow pl = n\pi\ \ (n = 1, 2, ...)$

$$\therefore \boxed{y = \cos\left(\frac{n\pi x}{l}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{l}\right) + \beta_n \sin\left(\frac{n\pi ct}{l}\right)\right\}}$$

[Alternately this can be written in the form

$$y = R_n \cos\left(\frac{n\pi x}{l}\right)\cos(\text{or}\sin)\left(\frac{n\pi ct}{l} + \phi_n\right)]$$

7.

- $$\left.\begin{array}{l} u_{tt} = c^2 u_{xx} \ (0 \leqslant x \leqslant l, t \geqslant 0) \\ u(0,t) = 0,\ u(l,t) = a\sin\omega t \\ u(x,0) = 0,\ \dot{u}(x,0) = 0 \end{array}\right\} \qquad (*)$$

Write $(x,t) = v(x,t) + a\sin(\omega t)\dfrac{\sin(\omega x/c)}{\sin(\omega l/c)}$

Then $(u_{tt} - c^2 u_{xx}) = 0 = (v_{tt} - c^2 v_{xx}) - a\omega^2 \quad (o)$

$\therefore \boxed{v_{tt} - c^2 v_{xx} = 0}$ (A)

$u(0,t) = 0 \Rightarrow \boxed{v(0,t) = 0}$; $u(l,t) = a\sin(\omega t) \Rightarrow \boxed{v(l,t) = 0}$ (B)

$u(x,0) = 0 \Rightarrow \boxed{v(x,0) = 0}$; $\dot{u}(x,0) = 0 \Rightarrow \boxed{\dot{v}(x,0) = -a\omega\dfrac{\sin(\omega x/c)}{\sin(\omega l/c)}}$ (C)

- Now - see e.g. $\boxed{2}$ in §2.2 of Notes - (A), (B) and first of (C) are satisfied by:

$$v(x,t) = \sum_{p=1}^{\infty} a_p \sin\left(\frac{p\pi x}{l}\right)\sin\left(\frac{p\pi ct}{l}\right)$$

- There remains the task of satisfying the second of (C), i.e.

$$\sum_{p=1}^{\infty} a_p \left(\frac{p\pi x}{l}\right)\sin\left(\frac{p\pi x}{l}\right) = -a\omega\frac{\sin\left(\frac{\omega x}{c}\right)}{\sin\left(\frac{\omega l}{c}\right)}$$

Using the standard technique, we multiply both sides by $\sin\left(\dfrac{q\pi x}{l}\right)$ and $\int_0^l dx$. This gives

$$a_q\left(\frac{q\pi c}{l}\right)\cdot\frac{l}{2} = -\frac{a\omega}{\sin(\omega l/c)}\int_0^l \sin\left(\frac{\omega x}{c}\right)\sin\left(\frac{q\pi x}{l}\right)dx$$

$$\therefore \ a_q = \frac{a\omega}{q\pi c\sin\left(\frac{\omega l}{c}\right)}\int_0^l \left[\cos\left(\frac{\omega}{c} + \frac{q\pi}{l}\right)x - \cos\left(\frac{\omega}{c} - \frac{q\pi}{l}\right)x\right]dx$$

$$= \frac{a\omega l}{q\pi c\sin\left(\frac{\omega l}{c}\right)}\left[\frac{\sin\left(\frac{\omega l}{c} + q\right)}{(q\pi c + \omega l)} + \frac{\sin\left(\frac{\omega l}{c} - q\pi\right)}{(q\pi c - \omega l)}\right]$$

$$= \frac{2a\omega c}{l}\frac{(-1)^q}{\left\{\left(\frac{q\pi c}{l}\right)^2 - \omega^2\right\}} \Rightarrow \boxed{\text{result given in question for u(x,t)}}$$

- When $\omega = \dfrac{\pi c}{l}$, the term $a\left\{\dfrac{\sin\left(\frac{\omega x}{c}\right)}{\sin\left(\frac{\omega l}{c}\right)}\right\}\sin\omega t$ and the term in the infinite series with $p = 1$ are undefined. Thus, for ω near $\dfrac{\pi c}{l}$, put

$$u(x,t) = u_*(x,t) + \frac{2ac}{l}\frac{(\pi c/l)}{(\pi c/l)^2}\sum_{p=2}^{\infty}\frac{(-1)^p \sin\left(\frac{p\pi x}{l}\right)\sin\left(\frac{p\pi ct}{l}\right)}{(p^2-1)}$$

$$\Rightarrow \boxed{u(x,t) = u_*(x,t) + \frac{2a}{\pi}\sum_{p=2}^{\infty}\frac{(-1)^p \sin\left(\frac{p\pi x}{l}\right)\sin\left(\frac{p\pi ct}{l}\right)}{(p^2-1)}}$$

where

$$u_*(x,t) = \lim_{\epsilon\to 0}\left\{\frac{a\sin\left[\frac{\pi x}{l}(1+\epsilon)\right]\sin\left[\frac{\pi ct}{l}(1+\epsilon)\right]}{\sin(\pi+\pi\epsilon)} - \frac{2ac}{l}\frac{\frac{\pi c}{l}(1+\epsilon)\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right)}{(\frac{\pi c}{l})^2[1-(1+\epsilon)^2]}\right\}$$

Now: (i) $\dfrac{1}{\sin(\pi+\pi\epsilon)} = -\dfrac{1}{\sin\pi\epsilon} \approx \boxed{-\dfrac{1}{\pi\epsilon}}$ (since $\sin x \approx x$ when $|x| \ll 1$)

(ii) $\dfrac{(1+\epsilon)}{[1-(1+\epsilon)^2]} = -\dfrac{1+\epsilon}{2\epsilon+\epsilon^2} = -\dfrac{(1+\epsilon)}{2\epsilon}(1+\frac{\epsilon}{2})^{-1} \approx -\dfrac{1}{2\epsilon}(1-\frac{\epsilon}{2})(1+\epsilon) \approx \boxed{-\dfrac{1}{2\epsilon}-\dfrac{1}{4}}$

(iii) $\sin\left[\frac{\pi x}{l}(1+\epsilon)\right] = \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\epsilon\pi x}{l}\right) + \sin\left(\frac{\epsilon\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right) \approx \boxed{\sin(\frac{\pi x}{l}) + \frac{\epsilon\pi x}{l}\cos(\frac{\pi x}{l})}$

(iv) likewise - see (iii) - $\sin\left[\frac{\pi ct}{l}(1+\epsilon)\right] \approx \boxed{\sin\left(\frac{\pi ct}{l}\right) + \frac{\epsilon\pi ct}{l}\cos\left(\frac{\pi ct}{l}\right)}$

- Thus $u_*(x,t) = \lim_{\epsilon\to 0}\left\{ -\frac{a}{\pi\epsilon}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right)\right.$

$$-\frac{a}{\pi\epsilon}\left(\frac{\epsilon\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right) - \frac{a}{\pi\epsilon}\left(\frac{\epsilon\pi ct}{l}\right)\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi ct}{l}\right)$$

$$\left. + \frac{a}{\pi\epsilon}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right) + \frac{a}{2\pi}\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi ct}{l}\right)\right\}$$

$$\Rightarrow \boxed{u_*(x,t) = \frac{a}{2\pi}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \frac{a}{l}\left\{x\cos\frac{\pi x}{l}\sin\frac{\pi ct}{l} + ct\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}\right\}}$$

(This can also be obtained in other ways, e.g. by L'Hopital's rule.)

N.B:

The terms in u_* in curly brackets are oscillations whose `amplitudes` increase `either` with x $\left[\text{viz. } \left(-\frac{ax}{l}\cos\frac{\pi x}{l}\right)\sin\frac{\pi ct}{l}\right]$ `or` with t $\left[\text{viz. } \left(-\frac{act}{l}\cos\frac{\pi ct}{l}\right)\sin\frac{\pi x}{l}\right]$. These are typical `resonant` behaviours caused (in this case) by `forcing` the end $x = l$ to oscillate at one of the natural frequencies of the system.