NB Questions 1 and 2 are relevant to resonance - see S2 Q7

1. The CF is $A\cos nt + B\sin nt$

   (i) $\omega \neq n$: Try PI $y = C\cos\omega t$. OK provided $C(-\omega^2 + n^2) = f \Rightarrow C = \dfrac{f}{n^2 - \omega^2}$

   $\therefore$ GS is $y = A\cos nt + B\sin nt + \dfrac{f}{(n^2-\omega^2)}\cos\omega t$

   $y(0) = \dot{y}(0) = 0 \Rightarrow B = 0,\ A = -\dfrac{f}{(n^2-\omega^2)}$ $\therefore$ $\boxed{y = \dfrac{f}{(n^2-\omega^2)}[\cos\omega t - \cos nt]}$

   (ii) $\omega = n$: Try PI $y = Dt\sin nt$ (Y1 or MAS221) $\Rightarrow \dot{y} = D\sin nt + Dnt\cos nt$,
   $\ddot{y} = 2Dn\cos nt - Dn^2 t\sin nt$.
   Satisfies ODE $\Rightarrow$ $2Dn\cos nt - Dn^2 t\sin nt + n^2\sin nt = f\cos nt \Rightarrow D = \dfrac{f}{2n}$
   $\therefore$ GS is $y = A\cos nt + B\sin nt + \dfrac{ft}{2n}\sin nt$

$$y(0) = \dot{y}(0) = 0 \Rightarrow A = B = 0 \ \therefore \ \boxed{y = \frac{ft}{2n}\sin nt}$$

The answer to (i), viz.

$$y = -f\frac{(\cos nt - \cos\omega t)}{(n^2-\omega^2)}$$

is not defined at $\omega = n$ but does it have a limit as $\omega \to n$? The answer is YES. The value of the limit can be found by l'Hopital's Rule or in the following way:

$$y = -f\frac{[-2\sin\frac{1}{2}(n-\omega)t \cdot \sin\frac{1}{2}(n+\omega)t]}{(n-\omega)(n+\omega)} \approx \frac{f\sin nt}{2n}\left\{\frac{\sin\frac{1}{2}(n-\omega)t}{\frac{1}{2}(n-\omega)}\right\}$$

Since

$$\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1,\ \lim_{\theta\to 0}\left\{\frac{\sin t\theta}{t\theta}\right\} = \lim_{\theta\to 0}\left\{\frac{\sin t\theta}{t\theta}\cdot t\right\} = t$$

Hence

$$\lim_{\omega\to n}\left\{\frac{\sin\frac{1}{2}(n-\omega)t}{\frac{1}{2}(n-\omega)}\right\} = t$$

and

$$\boxed{y \to \frac{ft}{2n}\sin nt \ \text{ as required, viz (ii)}}$$

NB the answer to (ii), when the forcing is at the natural frequency $\omega$, is an oscillation at a growing amplitude $\left(\dfrac{ft}{2n}\right)$. In practice, there is always damping (or friction), represented by the term $2\lambda\dot{y}$ in the ODE in Q2.

2. The CF is $A'e^{m_1 t}+B'e^{m_2 t}$ where $m_1,\ m_2$ are the roots of $m^2+2\lambda m+n^2=0 \Rightarrow m=-\lambda\pm i\sigma$ where $\sigma^2=n^2-\lambda^2$ [NB $\lambda << n \Rightarrow \sigma$ is real]. Thus CF is $e^{-\lambda t}A\cos\sigma t+B\sin\sigma t$

The PI is $\mathfrak{Re}\{\mathrm{C}^{i\omega t}\}$ where

$$\mathrm{C}\{-\omega^2+2\lambda i\omega+n^2\}=f \Rightarrow \mathrm{C}=\frac{f}{\{(n^2-\omega^2)+2\lambda i\omega\}}.$$

Now we can write $(n^2-\omega^2)+2\lambda i\omega=\mathfrak{Re}^{i\alpha}$ where

$$\begin{cases} R^2 &= (n^2-\omega^2)^2+4\lambda^2\omega^2 \\ R\cos\alpha &= (n^2-\omega^2) \\ R\sin\alpha &= 2\lambda\omega \end{cases}$$

$\Rightarrow \mathrm{C}=\dfrac{f}{R}e^{-i\alpha} \Rightarrow$ PI is $\dfrac{f}{R}\cos(\omega t-\alpha)$

Thus GS is $y=e^{-\lambda t}\{A\cos\sigma t+B\sin\sigma t\}+\frac{f}{R}\cos(\omega t-\alpha)$

$y(0)=\dot{y}(0)=0 \Rightarrow$ (after some algebra) $A=-\dfrac{f}{R}\cos\alpha,\ B=-\dfrac{f}{\sigma R}(\lambda\cos\alpha+\omega\sin\alpha)$

$$\therefore \boxed{y=\frac{f}{R}\left\{\cos(\omega t-\alpha)-e^{-\lambda t}\left[\cos\alpha\cos\sigma t+\left(\frac{\lambda\cos\alpha+\omega\sin\alpha}{\sigma}\right)\sin\sigma t\right]\right\}}$$

(i) (a) $\lambda=0,\ \omega\neq n \Rightarrow R=(n^2-\omega^2),\ \alpha=0,\ \sigma=n \ \Rightarrow$ result in Q1(i) above

(b) $\omega=n,\ \lambda\neq 0 \Rightarrow \alpha=\dfrac{\pi}{2},\ R=2\lambda n,\ \sigma=(n^2-\lambda^2)^{\frac{1}{2}} \Rightarrow$

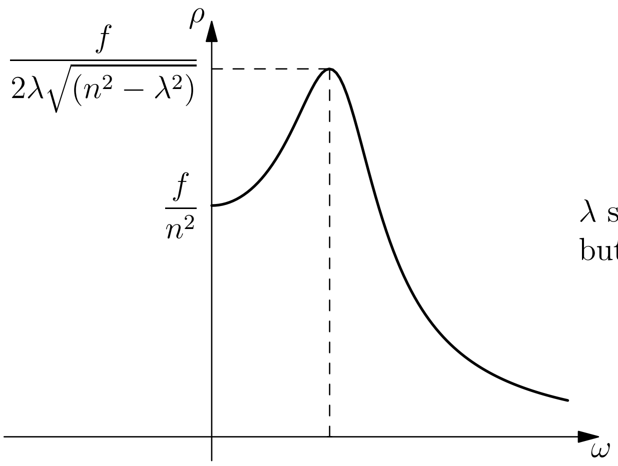
$$y=\frac{f}{2\lambda n}\left\{\sin nt-\frac{n}{(n^2-\lambda^2)^{\frac{1}{2}}}e^{-\lambda t}\sin(n^2-\lambda^2)^{\frac{1}{2}}t\right\}$$

As $\lambda\to 0$, this can be simplified to

$$\begin{aligned} y &\approx \frac{f}{2\lambda n}\{\sin nt\}\{1-e^{-\lambda t}\} \\ &\approx \frac{f\sin nt}{2n}\frac{(1-e^{-\lambda t})}{\lambda} \\ &\approx \frac{f\sin nt}{2n}\frac{[1-(1-\lambda t+t\lambda^2t^2...)]}{\lambda} \\ &\approx \frac{ft}{2n}\sin nt \ \boxed{\text{result in Q1(ii) above}} \end{aligned}$$

(ii) For large $\lambda t$, $y\approx\dfrac{f}{R}\cos(\omega t-\alpha)=\boxed{\rho\cos(\omega t-\alpha)}$. Here

$$\rho=\frac{f}{\{(\omega^2-n^2)^2+4\lambda^2\omega^2\}^{\frac{1}{2}}}$$




$\lambda$ small $\Rightarrow \rho_{max}$ is large.
but $\lambda \neq 0 \Rightarrow$ no singularity

3.

- With $\rho = \rho_0(1+s) = \rho_0 + \rho_0 s,\ u = \phi_x,\ v = \phi_y,\ w = \phi_z$, (3.13)
$\Rightarrow \rho_0 s_t = -\rho_0(\phi_{xx} + \phi_{yy} + \phi_{zz}) \Rightarrow \boxed{s_t = -(\phi_{xx} + \phi_{yy} + \phi_{zz})}$ (A)

- (3.11) $\dfrac{\partial}{\partial x}(\phi_t) = -c^2 s_x \Rightarrow \dfrac{\partial}{\partial x}(c^2 s + \phi_t) = 0.$
Likewise $\dfrac{\partial}{\partial y}(c^2 s + \phi_t) = \dfrac{\partial}{\partial z}(c^2 s + \phi_t) = 0.$
Thus $c^2 s + \phi_t = F(t)$.

  As $|\underline{x}| \to \infty$, there is no disturbance and conditions are steady. Thus $F(t) = 0$ so $\boxed{c^2 s = -\phi_t}$ (B)

- Eliminating $s$ between (A) and (B) $\Rightarrow c^2 s_t = -\phi_{tt} = -c^2(\phi_{xx} + \phi_{yy} + \phi_{zz})$

  $\Rightarrow \boxed{\phi_{tt} = c^2(\phi_{xx} + \phi_{yy} + \phi_{zz})}$

  (N.B. $(\phi_{xx} + \phi_{yy} + \phi_{zz})$ is often written $\nabla^2\phi$ ("delsquared $\phi$") and is known as the Laplacian of $\phi$.)

4. • $r^2 = x^2 + y^2 + z^2 \Rightarrow \dfrac{\partial}{\partial x}(r^2) = 2x \Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x$(chain rule)

$$\Rightarrow \boxed{\frac{\partial r}{\partial x} = r^{-1}x}$$

$$\therefore \frac{\partial \phi}{\partial x} = \frac{\partial \Phi}{\partial r}\frac{\partial r}{\partial x} = \boxed{r^{-1}\frac{\partial \Phi}{\partial r}x}$$

$$\& \ \frac{\partial^2 \phi}{\partial x^2} = \boxed{\frac{\partial}{\partial r}\left[r^{-1}\frac{\partial \Phi}{\partial r}\right] r^{-1}x^2 + r^{-1}\frac{\partial \Phi}{\partial r}}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 3r^{-1}\Phi_r + (x^2+y^2+z^2)r^{-1}\left\{r^{-1}\Phi_{rr} - r^2\Phi_r\right\} = \Phi_{rr} + 2r^{-1}\Phi_r$$

• Thus $c^2(\phi_{xx} + \phi_{yy} + \phi_{zz}) = \phi_{tt} \Rightarrow \boxed{c^2(\Phi_{rr} + 2r^{-1}\Phi_r) = \Phi_{tt}}$ (B)

• There are many ways of doing the last part. Here is one only.
Write
$r\Phi = \Psi \Rightarrow \Phi = r^{-1}\Psi \Rightarrow \Phi_r = r^{-1}\Psi_r - r^{-2}\Psi$
and
$\Phi_{rr} = r^{-1}\Psi_{rr} - 2r^{-2}\Psi_r + 2r^{-3}\Psi$.
Substitute into (B) to get
$c^2\left\{r^{-1}\Psi_{rr} - 2r^{-2}\Psi_r + 2r^{-3}\Psi + 2r^{-2}\Psi_r - 2r^{-3}\Psi\right\} = r^{-1}\Psi_{tt}$.
Hence $c^2\Psi_{rr} = \Psi_{tt}$. This is (1.6) in another notation, so the GS for $\Psi$ is $f(r-ct) + g(r+ct)$ by (1.12).
But $e^{i\omega/c(r-ct)}$ is $f(r-ct)$ with $f(\xi) = e^{i\omega\xi/c}$ and $e^{i\omega/c(r+ct)}$ is $g(r+ct)$ with $g(\xi) = e^{i\omega\xi/c}$.
Hence $\boxed{Ar^{-1}e^{i\omega/c(r \pm ct)} \text{ both satisfy (B)}}$ [NB See S1 Q6.]

5.

- Substitute $\phi = f(x)e^{i(\omega t - kz)}$ into $\phi_{xx} + \phi_{zz} = \dfrac{1}{c^2}\phi_{tt}$, obtaining $f'' - k^2 f = -\frac{\omega^2}{c^2} f \Rightarrow f'' + \left(\frac{\omega^2}{c^2} - k^2\right) f = 0.$

- Now suppose $k^2 > \dfrac{\omega^2}{c^2}$, and write $k^2 - \dfrac{\omega^2}{c^2} = m^2 \Rightarrow f'' - m^2 f = 0$
  $\Rightarrow f = Ae^{me} + Be^{-mx}$.
  Now $\dfrac{\partial \phi}{\partial x} = 0 \Rightarrow f' = 0$.
  Thus $f' = 0$ at $x = 0 \Rightarrow m(A - B) = 0 \Rightarrow B = A, \ \Rightarrow f = 2A\cosh mx$.
  Then $f' = 0$ at $x = d \Rightarrow 2A(\sinh md)m = 0 \Rightarrow A = 0 \Rightarrow \phi = 0$. No interest!

- If $k^2 = \dfrac{\omega^2}{c^2}$, we have $f'' = 0 \Rightarrow f = Bx + A^*$. $f' = 0$ at $x = 0, d \Rightarrow B = 0$. Thus $\boxed{f = 2Ae^{i(\omega t - kz)} \text{ and } k = k_0 = \dfrac{\omega}{c}}$ with $A^* = 2A$.

- If $k^2 < \dfrac{\omega^2}{c^2}$, write $\dfrac{\omega^2}{c^2} - k^2 = m^2 \Rightarrow f'' + m^2 f = 0 \Rightarrow f = Ae^{imx} + Be^{-imx}$.
  Then $f' = 0$ at $x = 0 \Rightarrow B = A, \ \Rightarrow f = 2A\cos mx$.
  Then $f' = 0$ at $x = d \Rightarrow -2A(\sin md)m = 0 \Rightarrow md = n\pi$ $(n = 1, 2, ...)$

  Thus $\boxed{f = 2A\cos\left(\dfrac{n\pi x}{d}\right) e^{i(\omega t - kz)} \text{ with } k = k_n = \left(\dfrac{\omega^2}{c^2} - \dfrac{n^2\pi^2}{d^2}\right)^{\frac{1}{2}}.}$

  [NB For real waves to propagate we need real $\omega, k$. Thus, if (for example) $\omega$ is fixed, there are only a finite number of $k$ values, those with $n \leq n_{max}$, where $n_{max}$ is the largest positive integer for which $n_{max} \leq \left(\dfrac{\omega d}{\pi c}\right)$.]

6. • $\phi = g(r)e^{i(\omega t - kz)} \Rightarrow \dfrac{\partial \phi}{\partial x} = \dfrac{\partial g}{\partial r}\dfrac{\partial r}{\partial x} e^{i(\omega t - kz)}$

$r^2 = x^2 + y^2 \Rightarrow 2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = r^{-1}$ (see answer to Q4)

$\therefore \dfrac{\partial \phi}{\partial x} = \left\{(r^{-1}g')x\right\} e^{i(\omega t - kz)}$

& $\dfrac{\partial^2 \phi}{\partial x^2} = \left\{\dfrac{\partial}{\partial r}(r^{-1}g')r^{-1}x^2 + (r^{-1}g')\right\} e^{i(\omega t - kz)}$

$= \left\{r^{-2}g''x^2 - r^{-3}g'x^2 + r^{-1}g'\right\} e^{i(\omega t - kz)}$

$\therefore \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = \left\{g'' - r^{-1}g' + 2r^{-1}g'\right\} e^{i(\omega t - kz)}$

$= \left\{g'' + r^{-1}g'\right\} e^{i(\omega t - kz)}$

• $\therefore \quad c^2(\phi_{xx} + \phi_{yy} + \phi_{zz}) = \phi_{tt} \Rightarrow g'' + \dfrac{1}{r}g' - k^2 g = -\dfrac{\omega^2}{c^2}g \Rightarrow$

$$\boxed{g''(r) + \frac{1}{r}g'(r) + m^2 g(r) = 0 \text{ where } m^2 = \frac{\omega^2}{c^2} - k^2} \qquad (*)$$

• Write

$$mr = x \Rightarrow \frac{dg}{dr} = m\frac{dg}{dx},\ m\frac{d^2g}{dr^2} = m^2\frac{d^2g}{dx^2}.$$

Thus $(*)$ becomes

$$x^2\frac{d^2g}{dx^2} + x\frac{dg}{dx} + x^2 g = 0.$$

Since $g$ is bounded at $x = r = 0$, $g \propto J_0(x) = J_0(mr)$. [See §2.5 in Notes.]
We need $g' = 0$ at $r = a \Rightarrow J_0'(ma) = 0 \Rightarrow ma = \beta_n$ where $\beta_n$ is the $n$th zero of $J_0'(x) = 0$ with $\beta_0 = 0$ (see sketch on handout).

• Thus, finally, $\boxed{g(r) \propto J_0\left(\dfrac{\beta_n r}{a}\right)}$ and $\boxed{k = k_n = \left(\dfrac{\omega^2}{c^2} - \dfrac{\beta_n^2}{a^2}\right)^{\frac{1}{2}}}$

(N.B. as is Q5, for fixed $\omega$ there are only a finite number of $k$ values. )
(N.B. In a real stethoscope most of the acoustic energy is carried by the mode with $\beta_n = 0$.)

7.

- Look for separable solutions of the form $\phi = X(x)T(t)$.

  Substitute into the PDE $\Rightarrow c^2 X''T = X\ddot{T} \Rightarrow c^2\dfrac{X''}{X} = \dfrac{\ddot{T}}{T}$.

- In the normal way, each side is a constant which we take to be $-\omega^2$ since we seek time-harmonic solutions, i.e.

$$\frac{X''}{X} = -\frac{\omega^2}{c^2},\ \frac{\ddot{T}}{T} = -\omega^2 \Rightarrow X = A\cos\left(\frac{\omega x}{c}\right) + B\sin\left(\frac{\omega x}{c}\right)$$

$$T \propto \cos(\omega t + \epsilon)$$

- $\phi = \left\{A\cos\dfrac{\omega x}{c} + B\sin\dfrac{\omega x}{c}\right\}\cos(\omega t+\epsilon)$

  $\phi_x = 0$ at $x = 0 \Rightarrow B = 0 \Rightarrow \phi = A\cos\left(\dfrac{\omega x}{c}\right)\cos(\omega t+\epsilon)$

  $\phi_t = 0$ at $x = l \Rightarrow \cos\left(\dfrac{\omega l}{c}\right) = 0 \Rightarrow \dfrac{\omega l}{c} = \left(n+\dfrac{1}{2}\right)\pi$

$$\Rightarrow \boxed{\omega = \omega_n = \frac{\pi c}{l}\left(n+\frac{1}{2}\right)} \text{ and } \boxed{\phi = \phi_n \propto \cos\left(\frac{\omega_n x}{c}\cos(\omega_n t + \epsilon_n)\right)}$$

  N.B. If you have done Q5 or Q6 you may ask why the cross-section of the tube does not appear in this question. This is because the work in this question relates to the $n = 0$ case in both Q5 and Q6 for which there is no dependence on cross-sectional position or shape.