1. We are given that

$$\frac{2\pi}{\omega} = \frac{60}{15} = 4s \Rightarrow \omega = \left(\frac{\pi}{2}\right) s^{-1}$$

From (4.19a), we find that

$$\omega^2 = gk = g^2/c \Rightarrow c = g/\omega$$

$$\therefore \quad c \approx \frac{9.8x2}{\pi} \text{ ms}^{-1} \approx 6.239 \text{ ms}^{-1}$$

1m $\approx$ 39.37 inches and 1 mile = 1760 yards = 1760 x 36 inches

$$\therefore \quad c \approx \frac{6.239 \text{ x } 39.37}{1760 \text{ x} 36} \text{ x } 3600 \text{ mph} \approx \boxed{14 \text{ mph}}$$

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2. By (4.20a) and (4.20b), the mean energy per unit area of the water surface is $(V_\rho + T_\rho) = \frac{1}{2}\rho g \eta_0^2$. For waves whose lateral extent is L, the mean energy per unit length in the direction of propagation is therefore $\boxed{\frac{1}{2}\rho g \eta_0^2 L}$

$$\frac{1}{2}\rho g \eta_0^2 L = \frac{1}{2} \text{x} \, 10^3 \, \text{x} \, 9.8 \, \text{x} \, 1.25^2 \, \text{x} \, 10^{-2} \, \text{x} \, 50 \, \text{kgms}^{-2} \approx \boxed{3.83 \, \text{x} \, 10^3 \text{ kgms}^{-2}}$$

By (4.16), $c^2 = \frac{g}{k} \tanh kh$ where $k = \frac{2\pi}{\lambda} = \frac{\pi}{5}$ m^{-1}

$$\therefore c^2 = \frac{5 \, \text{x} \, 9.8}{\pi} \tanh(0.8\pi) \text{ m}^2\text{s}^{-1} \approx 15.394 \text{ m}^2\text{s}^{-1} \Rightarrow \boxed{c \approx 3.92 \text{ ms}^{-1}}$$

The KE per unit length of the car is $\frac{1}{2}$ x 1200 x $3.92^2/3$ kgms^{-2} $\approx \boxed{3.07x10^3 \text{ kgms}^{-2}}$
The wave is more energetic than the car.

3.

- $\frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}$ at $z = 0 \Rightarrow f'(0) \sin kx \sin \omega t = -\eta_0 \omega \sin kx \sin \omega t$
  So (4.10) can be satisfied with given ϕ provided $\boxed{f'(0) = -\omega\eta_0}$ (1)

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \, \text{at} \, z = 0 \Rightarrow \omega f(0) \sin kx \cos \omega t + g\eta_0 \sin kx \cos \omega t = 0$$

So (4.11) can be satisfied with given ϕ provided $\boxed{f(0) = -\frac{g\eta_0}{\omega}}$ (2)

- There remains to satisfy $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial z^2}=0$ with $\dfrac{\partial\phi}{\partial z}=0$ at $z=-h$.
  $\therefore f=A\cosh k(z+h)$ exactly as for progressive waves.

$$(1)\Rightarrow kA\sinh kh=-\omega\eta_0\Rightarrow A=-\frac{\omega\eta_0}{k}\frac{1}{\sinh kh}\Rightarrow \boxed{f(z)=-\frac{\omega\eta_0}{k}\frac{\cosh k(z+h)}{\sinh kh}}$$

$$\text{Then (2)}\Rightarrow -\frac{\omega\eta_0}{k}\frac{\cosh kh}{\sinh kh}=-\frac{g\eta_0}{\omega}\Rightarrow\boxed{\omega^2=gk\tanh kh}\checkmark\Rightarrow\boxed{(4.16)\text{ holds}}$$

- Now $\dfrac{dX}{dt}\approx\left.\dfrac{\partial\phi}{\partial x}\right|_{(x_0,z_0)}$, $\dfrac{dZ}{dt}\approx\left.\dfrac{\partial\phi}{\partial z}\right|_{(x_0,z_0)}$, as for progressive waves (see §4.7)

$$\therefore\ \frac{dX}{dt}=-\omega\eta_0\frac{\cosh k(z_0+h)}{\sinh kh}\cos kx_0\sin\omega t\,,\ \frac{dZ}{dt}=-\omega\eta_0\frac{\sinh k(z_0+h)}{\sinh kh}\sin kx_0\sin\omega t$$

Taking X and Z to have zero time means $\Rightarrow$

$$\left.\begin{aligned}X&=\eta_0\frac{\cosh k(z_0+h)}{\sinh kh}\cos kx_0\cos\omega t\\ Z&=\eta_0\frac{\sinh k(z_0+h)}{\sinh kh}\sin kx_0\sin\omega t\end{aligned}\right\}\qquad\boxed{\Rightarrow Z=mx\,,\ m=\tanh k(z_0+h)\tan kx_0}$$

$\Longrightarrow$ $\boxed{\text{Each particle does SHM.}}$

4.
- With the given form of ϕ, (4.8) $\Rightarrow f''=k^2 f$ (i.e (4.14)).
  As in the Notes, the solution satisfying (4.9) is $f(z)=A\cosh k(z+h)$
- (4.10) $\Rightarrow\dfrac{\partial\eta}{\partial t}=\dfrac{\partial\phi}{\partial z}$ at $z=0\Rightarrow-\omega\eta_0\cos(kx-\omega t)=kA\sinh kh\cos(kx-\omega t)$
  $\therefore\ A=-\dfrac{\omega\eta_0}{k\sinh kh}$ and $\phi=-\dfrac{\omega\eta_0}{k}\dfrac{\cosh k(z+h)}{\sinh kh}\cos(kx-\omega t)$
- $(*)\Rightarrow-\dfrac{\omega^2\eta_0}{k}\coth kh+g\eta_0=-\dfrac{Tk^2\eta_0}{\rho}$ [cancelling $\sin(kx-\omega t)$]

$$\Rightarrow\frac{\omega^2}{k\tanh kh}=\left(g+\frac{Tk^2}{\rho}\right)\Rightarrow\boxed{\omega^2=gk\tanh kh\left(1+\frac{Tk^2}{\rho g}\right)}$$

- Now take $\tanh kh=1\Rightarrow\omega^2=gk(1+x^2)$

$$c=\frac{\omega}{k}\left\{\frac{g}{x}\left(\frac{T}{\rho g}\right)^{\frac12}(1+x^2)\right\}^{\frac12}=\boxed{\left(\frac{gT}{\rho}\right)^{\frac14}\left(\frac{1+x^2}{x}\right)^{\frac12}}$$

$\omega^2=gk(1+x^2)$ and $c_g=\dfrac{d\omega}{dk}\Rightarrow 2\omega c_g=g(1+x^2)+gk.2x\dfrac{dx}{dk}$

But $\dfrac{dx}{dk}=\left(\dfrac{T}{\rho g}\right)^{\frac12}$ so $c_g=g\dfrac{(1+3x^2)}{2\omega}=\dfrac{c}{2}\dfrac{g}{kc^2}(1+3x^2)$

Now $\quad kc^2 = g\left(\frac{Tk^2}{\rho g}\right)^{\frac{1}{2}}\left(\frac{1+x^2}{x}\right) = g(1+x^2)$

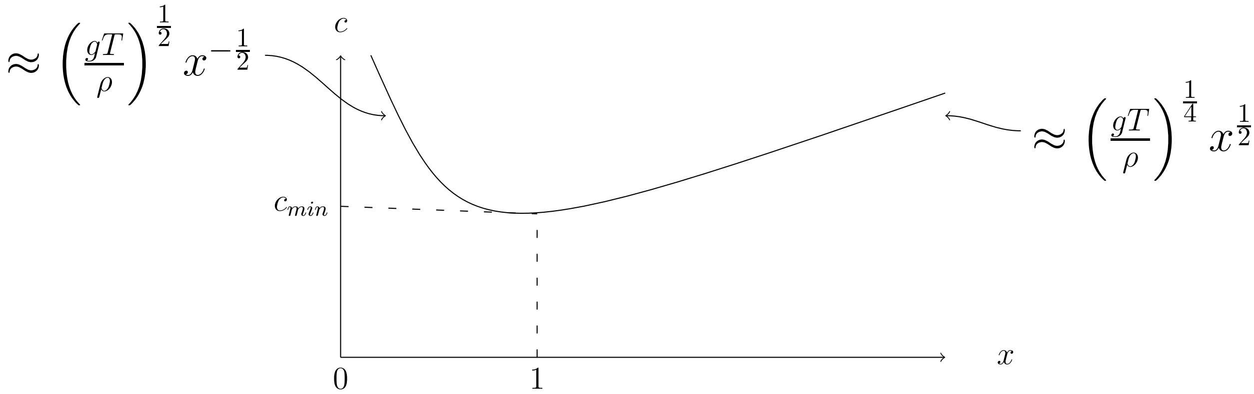
$$\therefore \boxed{c_g = \frac{c}{2}\left(\frac{1+3x^2}{1+x^2}\right)}$$

- c has a minimum when $\quad \frac{d}{dx}\left(\frac{1+x^2}{x}\right) = 0 \Rightarrow 2x^2 = 1+x^2 \Rightarrow \boxed{x=1}$

When $x = 1,\ c = c_{min} = \sqrt{2}\left(\frac{gT}{\rho}\right)^{\frac{1}{4}}$ (‡)

Now $\quad \frac{c_g}{c} = \frac{1}{2}\frac{(1+3x^2)}{(1+x^2)} \Rightarrow \frac{c_g}{c} - 1 = \frac{1}{2}\frac{(x^2-1)}{x^2+1}$

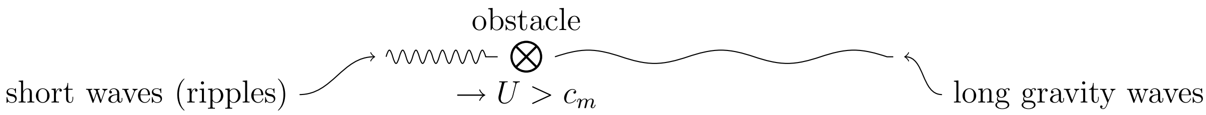
Thus $\boxed{c_g > c \text{ for } x > 1,\ c_g < c \text{ for } x < 1}$ and $c_g = c$ for $x = 1$



$$c_{min} = \sqrt{2}\left(\frac{9.8 \,\mathrm{x}\, 7.4 \,\mathrm{x}\, 10^{-2}}{10^3}\right)^{\frac{1}{4}} \mathrm{ms}^{-1} \approx \boxed{2.32 \,\mathrm{x}\, 10^{-1} \mathrm{ms}^{-1}}$$

5. (i) A fixed obstacle in a stream of speed U is equivalent to an obstacle moving with speed $-U$ in a stationary stream. The only waves that can keep up with the obstacle and that appear steady when viewed from the obstacle must evidently satisfy $c(k) = U$.

(ii) From the sketch of c against k (see above), for any value of $U > c_{min}$ there are two possible values of k for which $c(k) = U$.
For the larger value of k (shorter waves) $c_g > c = U$ from work in Q4. Hence these short waves, as a group, are upstream of the obstacle (this can easily be observed).



(iii) $U < c_{min}$ means no steady waves are possible.

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6. When we impose the velocity $-v$, we obtain a situation in which the observer is at rest, the source has velocity $-v$ and the waves travel with speed $c - v$.
We apply (4.32) to get

$$n'' = n(c-v)/[(c-v)-(-v)] = \boxed{n(c-v)/c}.$$

Likewise, impose the velocity $-v$ again, obtaining a situation in which the observer is at rest, the source has velocity $u - v$ and the waves travel with speed $c - v$.
Again apply (4.32) to get

$$n''' = n(c-v)/\{(c-v)-(u-v)\} = \boxed{n(c-v)/(c-u)}.$$

We have, using (4.32) : $272 = n\dfrac{c}{c-u}$, $256 = n\dfrac{c}{c+u}$

$$\therefore \frac{272}{256} = \frac{c+u}{c-u} \Rightarrow 16c = 528u \Rightarrow \boxed{u = \frac{c}{33}}.$$

• Now we need $\dfrac{n(c+v)}{c-u} = 280$ and, from above, $\dfrac{n}{(c-u)} = \dfrac{272}{c}$

$$\therefore 272\left(1+\frac{v}{c}\right) = 280 \Rightarrow 1 + \frac{v}{c} = \frac{280}{272} = \frac{35}{34} \Rightarrow \boxed{v = \frac{c}{34}}$$
