1. The associated equations are:

$$\frac{dx}{y} = -\frac{dy}{2xy} = \frac{dz}{2xz}$$

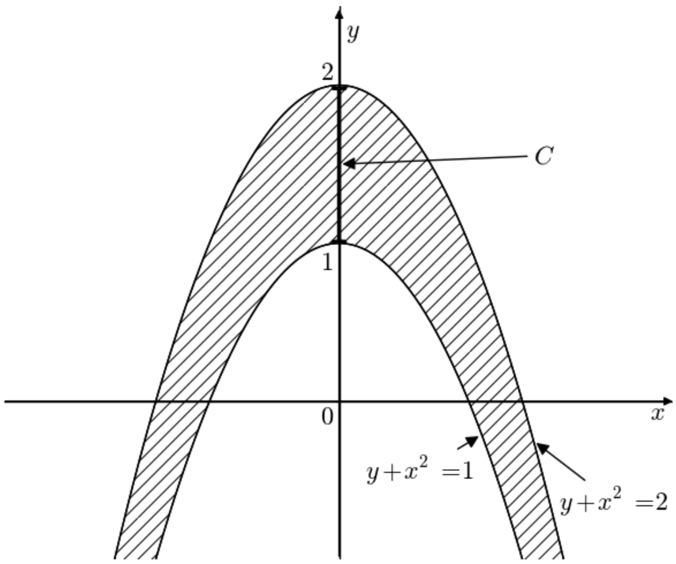
$\therefore \quad \frac{dy}{dx} = -2x \Rightarrow y + x^2 = \alpha$ <u>and</u> $\frac{dz}{z} = -\frac{dy}{y} \Rightarrow \ln z = -\ln y + \beta'$

$\Rightarrow z = \frac{\beta}{y}$

On $x = 0$, $1 \leq y \leq 2$, $y = \alpha$, $z = \frac{\beta}{\alpha} = \alpha^3 \quad \Rightarrow \quad \beta = \alpha^4$

$\therefore \boxed{z = \frac{(y + x^2)^4}{y}}$

Data prescribed on C:
$x = 0$ and $1 \leq y \leq 2 \Rightarrow$ solution defined only on region bounded by characteristics through $(0, 1)$ and $(0, 2) \Rightarrow$ hatched region



2. The associated equations are: $\dfrac{dx}{x^3} = -\dfrac{dy}{1} = \dfrac{du}{0}$

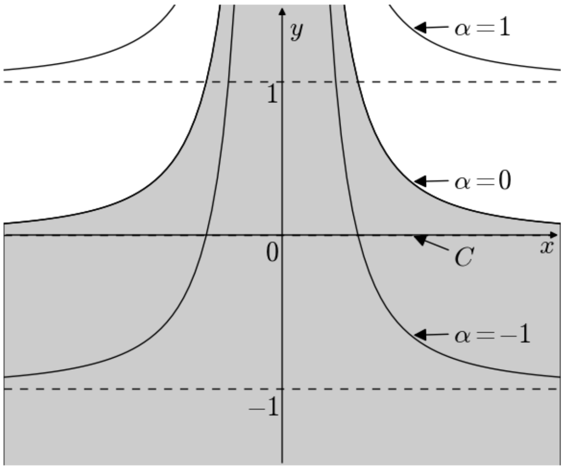
$\therefore \quad -\dfrac{1}{2x^2} = -y + \alpha \Rightarrow y - \dfrac{1}{2x^2} = \alpha$ <u>and</u> $u = \beta$

On $y = 0$, $\alpha = -\dfrac{1}{2x^2}$ (all x) $\Rightarrow x^2 = -\dfrac{1}{2\alpha}$ and $u = \dfrac{1}{1+x^2} = \beta$

$$\therefore \quad \beta = \frac{2\alpha}{2\alpha - 1} \Rightarrow u = \frac{2\left(y - \frac{1}{2x^2}\right)}{2\left(y - \frac{1}{2x^2} - 1\right)} = \boxed{\frac{(1 - 2x^2 y)}{(1 + x^2 - 2x^2 y)}}$$

The solution is defined only on the region bounded by the characteristics crossing C: $y = 0$, $-\infty < x < \infty$

$\boxed{\text{Thus it is not defined when } y \geq \dfrac{1}{2x^2}}$



3. In $x = \xi + F(\xi)t$, regard ξ as a function of (x, t). Thus

$$\frac{\partial}{\partial t} \Rightarrow 0 = \xi_t + F(\xi) + F'(\xi)\xi_t \cdot t \Rightarrow \boxed{\xi_t = -\frac{F(\xi)}{1+F'(\xi)t}}$$

$$\frac{\partial}{\partial x} \Rightarrow 1 = \xi_x + F'(\xi)\xi_x t \Rightarrow \boxed{\xi_x = \frac{1}{1+F'(\xi)t}}$$

(a) If $\rho = f(\xi)$ on the characteristics then $\boxed{\rho(x,0) = f(x)}$ since $x = \xi$ when $t = 0$. ✔
Also:

$$\begin{aligned} \rho_t + c(\rho)\rho_x &= f'(\xi)\xi_t + F(\xi)f'(\xi)\xi_x \\ &= \frac{f'(\xi)}{1+F'(\xi)t}\{-F(\xi) + F(\xi)\} = \boxed{0} \end{aligned}$$ ✔

(b) ξ_x and ξ_t are undefined (hence solution breaks down) if there is a $t > 0$ for which $1 + F'(\xi)t = 0 \Rightarrow$ $\boxed{\text{breakdown if } \exists\ \xi \text{ with } F'(\xi) < 0.}$

N.B. The result of (b) agrees with (5.27) and (5.28) in Notes obtained by another method.

4. $q(\rho) = \rho v(\rho) = \left(\frac{V}{P}\right)(\rho P - \rho^2)$ by (5.12)

$\therefore \quad q(\rho) = \frac{V}{P}\left\{\frac{P^2}{4} - \left(\rho - \frac{P}{2}\right)^2\right\}$ on completing the square.

$\therefore \quad q_m = \frac{1}{4}VP \Rightarrow P \approx \frac{4 \times 7400}{82} \approx \boxed{\text{360 cars per mile}}$

$c(\rho) = V\left(1 - \frac{2\rho}{P}\right)$ so when $\rho \approx 200$ cars per mile, $c \approx -\frac{1}{9}V \approx \boxed{-9\text{mph}}$

5. $\dfrac{\partial c}{\partial t} = c'(\rho)\rho_t \qquad \dfrac{\partial c}{\partial x} = c'(\rho)\rho_x$

$\therefore \quad c_t + cc_x = c'(\rho)\left[\rho_t + c(\rho)\rho_x\right] = \boxed{0}$ ✔

$\dfrac{dt}{1} = \dfrac{dx}{c} = \dfrac{dc}{0} \Rightarrow c = \text{constant on lines:} \quad \dfrac{dx}{dt} = c$

$\rho(x,0) = f(x)$ at $t = 0 \Rightarrow c = c\{f(\xi)\} = F(\xi)$ when $x = \xi$ at $t = 0$

$\therefore \quad \boxed{c = F(\xi) \text{ on straight lines } x = \xi + F(\xi)t}$

We know that:

$$U = \frac{q(\rho_1) - q(\rho_2)}{\rho_1 - \rho_2}$$

(a) When:

$$q(\rho) = \alpha\rho^2 + \beta\rho + \gamma, \quad \frac{q(\rho_1) - q(\rho_2)}{(\rho_1 - \rho_2)} = \frac{\alpha(\rho_1^2 - \rho_2^2) + \beta(\rho_1 - \rho_2)}{(\rho_1 - \rho_2)}$$

$\therefore \quad U = \alpha(\rho_1 + \rho_2) + \beta.$

But $c(\rho) = 2\alpha\rho + \beta$, so $c_1 + c_2 = 2\alpha(\rho_1 + \rho_2) + 2\beta \Rightarrow \boxed{U = \frac{1}{2}(c_1 + c_2)}$

(b) For weak shocks, expand $q(\rho)$ in a Taylor series, so $q(\rho) \approx q(\rho_1) + q'(\rho_1)(\rho - \rho_1) + \frac{1}{2}q''(\rho_1)(\rho - \rho_1)^2$ and this is a good approximation if $|(\rho_2 - \rho_1)|/(\rho_1 + \rho_2) \ll 1$. Thus $q(\rho) = \alpha\rho^2 + \beta\rho + \gamma \quad \left[\alpha = \frac{1}{2}q''(\rho_1),\ \beta = q'(\rho_1) - \rho_1 q''(\rho_1),\ \gamma = q(\rho_1) - \rho_1 q_1 + \frac{1}{2}\rho_1^2 q''(\rho_1)\right]$ and $\boxed{\text{(a) holds.}}$

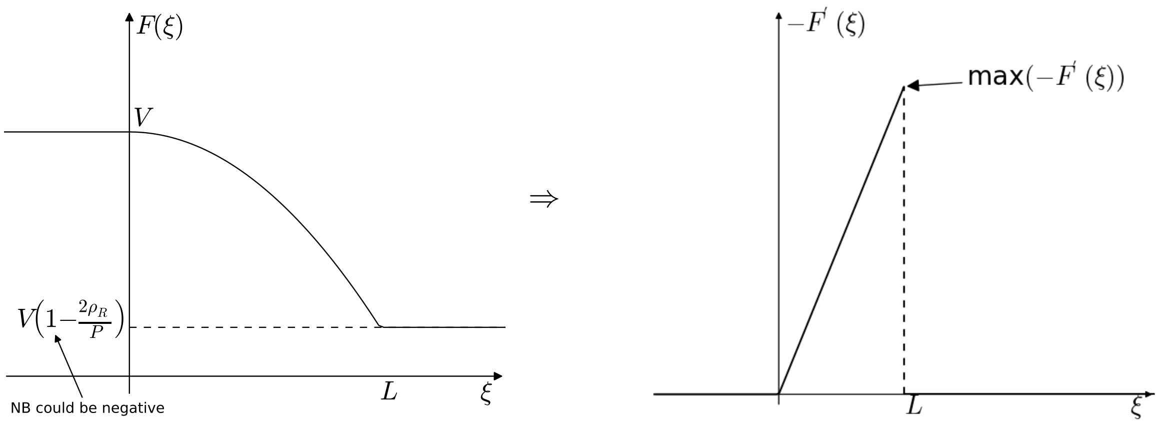
6. We have:

$$c(\rho) = \frac{d}{d\rho}\rho v(\rho) = \frac{V}{P}(P - 2\rho).$$

Thus

$F(\xi) = c\{\rho(\xi, 0)\} = V (\xi \leq 0),\ V\left\{1 - \frac{2\rho_R \xi^2}{PL^2}\right\}\ (0 \leq \xi \leq L),\ V\left\{1 - \frac{2\rho_R}{P}\right\}\ (\xi \geq L).$

The solution is $\rho = \rho(\xi, 0) = f(\xi)$ on $x = \xi + F(\xi)t$



- According to the theory- see (5.27) or (5.28) in Notes- the solution first breaks down for $t = T_{min} = \dfrac{1}{max\{-F'(\xi)\}}$.

  This is for $\xi = L$ when $F'(\xi) = -\dfrac{4\rho_R V}{PL} \Rightarrow \boxed{t = T_{min} = \dfrac{PL}{4\rho_R V}}\ x = L + V\left(1 - \dfrac{2\rho_R}{P}\right)\dfrac{PL}{4\rho_R V}$

  $\Rightarrow x = L + \dfrac{PL}{4\rho_R} - \dfrac{L}{2} = \boxed{\dfrac{L}{2} + \dfrac{P}{4\rho_R} \cdot L}$