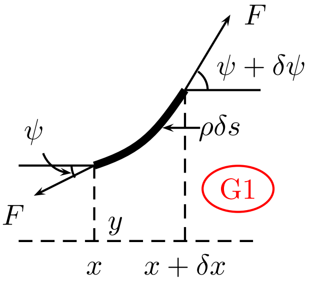
Answers and Mathscheme

1.



G1

$N2 \Rightarrow F\delta\psi \approx \rho\delta s\dfrac{\partial^2 y}{\partial t^2}$ M1A1

But oscillations small $\Rightarrow \delta s \approx \delta x,\ \psi \approx \dfrac{\partial y}{\partial x}$

$\Rightarrow \delta\psi \approx \dfrac{\partial^2 y}{\partial x^2}\delta x \Rightarrow$ $y_{xx} = \dfrac{1}{c^2}y_{tt}$ M1A1

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- Write $u = x - ct,\ v = x + ct$ M1

$y_x = y_u + y_v \Rightarrow y_{xx} = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)(y_u + y_v) = y_{uu} + 2y_{uv} + y_{vv}$

M2A1+A1

$y_t = -cy_u + cy_v \Rightarrow y_{tt} = c^2\left(\dfrac{\partial}{\partial u} - \dfrac{\partial}{\partial v}\right)(y_u - y_v) = c^2(y_{uu} - 2y_{uv} + y_{vv})$

- Substitute in PDE $\Rightarrow y_{uv} = 0$ M1A1 VERIFICATION M1A2 ONLY

$y_{uv} = 0 \Rightarrow y_u = f_*(u)$ M1A1 $\Rightarrow y = \displaystyle\int^u f_*(s)ds + g(v)$

11 ↥ *BW* $\Rightarrow$ $y(x,t) = f(x - ct) + g(x + ct)$ M1A1

- $y = f(x - ct) + g(x + ct)$

$t = 0 : f(x) + g(x) = 0$ M1A1 and $-cf'(x) + cg'(x) = V\cos 2kx$

$\Rightarrow -f(x) + g(x) = \dfrac{V}{2kc}\sin 2kx\ (+d) \rightarrow$ no effect on y M1A1

$\therefore f(x) = -\dfrac{V}{4kc}\sin 2kx, g(x) = \dfrac{V}{4kc}\sin 2kx$ A1 ← FOR EITHER

$\therefore$ $y(x,t) = \dfrac{V}{4kc}\{\sin[2k(x + ct)] - \sin[2k(x - ct)]\}$ *M1A1* FT to end bits

$y(x,t) = \dfrac{V}{2kc}\cos(2kx)\sin(2kct) \Rightarrow y(x,t) = 0$ when $2kx = (n + \dfrac{1}{2})\pi$

$\Rightarrow$ $x = \dfrac{(2n+1)\pi}{4k}$ A1

9 Standard waves with wavelength $\dfrac{\pi}{k}$ and period $\dfrac{\pi}{kc}$ A1

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2. Let $y_n = \sin(\frac{n\pi x}{l})\cos(\frac{n\pi ct}{l})$ M1A1 M1A1

Then $c^2 y_n'' - \ddot{y}_n = -\frac{n^2\pi^2c^2}{l^2}y_n - (-\frac{n^2\pi^2c^2}{l^2})y_n = 0$ so linearity $\Rightarrow$ PDE satisfied

(a) ok since $y_n(0,t) = y_n(l,t) = 0$ M1A1

(c) ok since $\dot{y}_n|_{t=0} = 0$ because $\sin(\frac{n\pi ct}{l}) = 0$ at $t = 0$ M1A1

$\boxed{8}$ ↑ BW

M1A1 M1A1

We need $\frac{h}{l^2}x(l-x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \Rightarrow \frac{a_n l}{2} = \int_0^l \frac{h}{l^2}x(l-x)\sin\frac{n\pi x}{l}dx$

$$\Rightarrow a_n = \frac{2h}{l^3}\Big[\underbrace{\cancel{\Big[-\frac{l}{n\pi}\cos(\frac{n\pi x}{l})x(l-x)\Big]_0^l}}_{=0} + \frac{l}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}dx\Big]$$ $M1A1$

$$= \frac{2h}{n\pi l^2}\Big[\underbrace{\cancel{\Big[\frac{l}{n\pi}\sin(\frac{n\pi x}{l})(l-2x)\Big]_0^l}}_{=0} + \frac{2l}{n\pi}\int_0^l \sin\frac{n\pi x}{l}dx\Big]$$

$$= \frac{4h}{(n\pi)^3}\Big[\cos(\frac{n\pi x}{l})\Big]_0^l = \begin{cases} 8h/(n\pi)^3 & \text{(n odd)} \\ 0 & \text{(n even)} \end{cases}$$ $M1A1 + A1$

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$$V = \frac{1}{2}\rho c^2 \int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}\rho c^2 \frac{\pi^2}{l^2}\sum_{n=0}^{\infty}\cos^2\frac{n\pi ct}{l}n^2\int_0^l a_n^2\cos^2\frac{n\pi x}{l}dx$$ $M1A1 + A1$

$M1A1$ if M0 here, no more maths

by orthogonality

$\boxed{8}$

$$\Rightarrow V = \frac{\rho\pi^2c^2}{4l}\sum_{n=0}^{\infty}a_n^2 n^2\cos^2\frac{n\pi ct}{l} = \frac{16\rho h^2c^2}{\pi^4 l}\sum_{m=0}^{\infty}\frac{\cos^2\{\frac{(2m+1)\pi ct}{l}\}}{(2m+1)4}$$

similar examples seen. M1A1 M1

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3.

(i) $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}\Big((\rho_0 + [\rho - \rho_0])u\Big) + \dfrac{\partial}{\partial y}\Big((\rho_0 + [\rho - \rho_0])v\Big) + \dfrac{\partial}{\partial z}\Big((\rho_0 + [\rho - \rho_0])w\Big) = 0$ M1/A1

$\dfrac{\partial \rho}{\partial t} + \rho_0\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) + \dfrac{\partial}{\partial x}\underbrace{[(\rho - \rho_0)u + \cdots]}_{\text{2nd order in smallness}} = 0$ M1A1

$\Rightarrow \rho_t + \rho_0(u_x + v_y + w_z) \approx 0$ A1

$\boxed{5}$

(ii) $\rho_{tt} = -\rho_0\dfrac{\partial}{\partial x}(u_t) - \rho_0\dfrac{\partial}{\partial y}(v_t) - \rho_0\dfrac{\partial}{\partial z}(w_t)$ A1/M1

N2 for each component M1/A1

$\rho_0 u_t = -p_x, \quad \rho_0 v_t = -p_y, \quad \rho_0 w_t = -p_z$

$\Rightarrow \rho_{tt} = p_{xx} + p_{yy} + p_{zz}$ A1

But $p = f(\rho) \Rightarrow (p - p_0) = \dfrac{df}{d\rho}|_{\rho_0}(\rho - \rho_0) = c^2(\rho - \rho_0)$ M2/A1

$\Rightarrow p_{xx} = c^2\rho_{xx},$ A1 $p_{yy} = c^2\rho_{yy}, \quad p_{zz} = c^2\rho_{zz}$

$\Rightarrow \rho_{tt} = c^2(\rho_{xx} + \rho_{yy} + \rho_{zz})$ A1

$\boxed{10}$

(iii) $\rho = \rho_0(1 + s), \ u = \phi_x, \ v = \phi_y, \ w = \phi_z$

$\Rightarrow \rho_0 s_t = -\rho_0(\phi_{xx} + \phi_{yy} + \phi_{zz})$

$s_t = -(\phi_{xx} + \phi_{yy} + \phi_{zz}) \qquad (1)$ M1/A1

From N2

$\dfrac{\partial}{\partial x}(\phi_t) = -c^2 s_x \overset{\text{M1}}{\Rightarrow} \dfrac{\partial}{\partial x}(c^2 s + \phi_t) = 0 \overset{\text{M1}}{\Rightarrow} c^2 s + \phi_t = F(t)$ A1

As $|x| \to \infty$, $\nexists$ disturbance $\Rightarrow F(t) = 0$ M1/A1

$$\Rightarrow c^2 s = -\phi_t \qquad (2)$$ A1

From (1) & (2)

$c^2 s_t = -\phi_{tt} = -c^2(\phi_{xx} + \phi_{yy} + \phi_{zz})$ M1

$\Rightarrow \phi_{tt} = c^2(\phi_{xx} + \phi_{yy} + \phi_{zz})$ A1

$\boxed{10}$

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4. (i)

(a): vertical component of velocity approaches zero at great depth. M2

(b): vertical component of velocity of free surface is consistent with that prescribed by η. M2

(c): pressure is continuous at free surface. M2

$\boxed{6}$

(ii) Try

$\phi = f(z)\sin(kx - \omega t)$ to be consistent with (ii) M1/A1

$f'' - k^2 f = 0$ A1

$f = Ae^{kz} + Be^{-kz}$ A1/M1

(a) $\Rightarrow B = 0$ A1, $\phi = Ae^{kz}\sin(kx - \omega t)$ A1

A1

(b) $\Rightarrow Ak\sin(kx - \omega t) = a\omega\sin(kx - \omega t) \Rightarrow A = \frac{a\omega}{k}$ A1

$\phi = \frac{a\omega}{k}e^{kz}\sin(kx - \omega t)$ A1

(c) $-\frac{a\omega^2}{k}\cos(kx - \omega t) + ga\cos(kx - \omega t) = 0$ A1/M1

$\Rightarrow \omega^2 = gk$ A1

$\boxed{13}$

$c = \frac{\omega}{k} = \sqrt{\frac{g}{q}}$ M1/A1

$c_g = \frac{d\omega}{dk} = \frac{d}{dk}\sqrt{\frac{g}{k}} = \frac{1}{2}\sqrt{\frac{g}{k}}$ M1/A1 $\Rightarrow c_g = \frac{1}{2}c.$ A1

Energy is always propagating with c_g. M1

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5. (i) The associated equations are $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xy}$ M1A1

$\therefore xdx = ydy \Rightarrow x^2 - y^2 = \alpha$ M1A1

Then $\dfrac{dz}{dx} = x \Rightarrow z = \dfrac{1}{2}x^2 + \beta$ M1A1

Hence $z - \dfrac{1}{2}x^2$ is a function of $(x^2 - y^2) \Rightarrow$ GS is $z = \dfrac{1}{2}x^2 + F(x^2 - y^2)$ M1A1

For $x^2 \geqslant y^2, z = e^{-x^2}, y = 0$

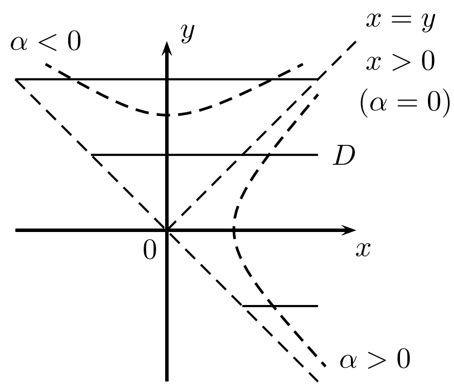
$\Rightarrow F(x^2) = e^{-x^2} - \dfrac{1}{2}x^2$ M2A1

$\Rightarrow$ $z = \dfrac{1}{2}y^2 + e^{-(x^2-y^2)}$ $(x^2 \geqslant y^2)$

For $x^2 \leqslant y^2, z = e^{-y^2}, x = 0$ needed for A1

13 $\Rightarrow F(-y^2) = e^{-y^2}$ M1A1 $\leftarrow$ In either order

$\Rightarrow$ $z = \dfrac{1}{2}x^2 + e^{(x^2-y^2)}$ $(x^2 \leqslant y^2)$ B1



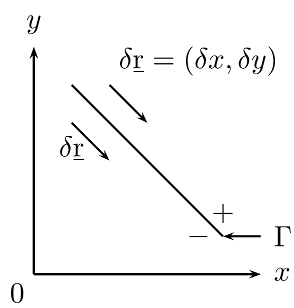
(ii) We note that either approach gives $z = \dfrac{1}{2}x^2 + 1$ on $x = y, x > 0$ $(*)$
The characteristics that cross $x = 0, y \geqslant 0$ and $y = 0, x \geqslant 0$ cannot be extended across $x + y = 0$, so D is $x + y \geqslant 0$ $\leftarrow$ M1A1

$(*)$ shows z is continuous everywhere in D. M1 B1

The only curve on which discontinuities in z_x, z_y can exist is $y = x\ (x \geqslant 0)$

By calculation we find that as $x \to y$ in $x^2 > y^2, z_x \to -2x, z_y \to 3x$, and that as $x \to y$ in $x^2 < y^2, z_x \to 3x, z_y \to -2x$.

7 $\Rightarrow$ jumps in z_x, z_y of magnitudes $\pm 5x$ M1A1



(iii) Next:

In $+$ $\delta z^+ = \dfrac{\partial z^+}{\partial x}\delta x + \dfrac{\partial z^+}{\partial y}\delta y,\ P\dfrac{\partial z^+}{\partial x} + Q\dfrac{\partial z^+}{\partial y} = R$

BW

In $-$ $\delta z^- = \dfrac{\partial z^-}{\partial x}\delta x + \dfrac{\partial z^-}{\partial y}\delta y,\ P\dfrac{\partial z^-}{\partial x} + Q\dfrac{\partial z^-}{\partial y} = R$

Since z is cont., $\delta z^+ = \delta z^-$ $M1A1 + A1$

$\therefore \delta x\left[\dfrac{\partial z}{\partial x}\right]_-^+ + \delta y\left[\dfrac{\partial z}{\partial y}\right]_-^+ = 0$ and $P\left[\dfrac{\partial z}{\partial x}\right]_-^+ + Q\left[\dfrac{\partial z}{\partial x}\right]_-^+ = 0$

5 The necessary condition for $\left[\dfrac{\partial z}{\partial x}\right]_-^+$ and $\left[\dfrac{\partial z}{\partial x}\right]_-^+$ to be non-zero is

$\dfrac{\delta x}{\delta y} = \dfrac{P}{Q} \Rightarrow$ $\dfrac{dy}{dx} = \dfrac{Q}{P}$ M1A1

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