Answers and Mathscheme

1. • Write $u = x - ct$, $v = x + ct$ (M1) $f(x-ct)+g(x+ct)$ is a solution $\Rightarrow$ max 6/11

$$\therefore \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v} \Rightarrow \frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial\phi}{\partial u} + \frac{\partial\phi}{\partial v}\right)$$
$$= \phi_{uu} + 2\phi_{uv} + \phi_{vv}$$

and $$\frac{\partial\phi}{\partial t} = -c\frac{\partial\phi}{\partial u} + c\frac{\partial\phi}{\partial v} \Rightarrow \frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial\phi}{\partial u} - \frac{\partial\phi}{\partial v}\right)$$ (M2A1+A1)
$$= c^2(\phi_{uu} - 2\phi_{uv} + \phi_{vv})$$ FT if sensible

• Substitute in PDE $\Rightarrow$
$\phi_{uu} + 2\phi_{uv} + \phi_{vv} = \phi_{uu} - 2\phi_{uv} + \phi_{vv} \Rightarrow \phi_{uv} = 0$ (M1A1)

• $\phi_{uv} = 0 \Rightarrow \phi_u = f_*(u)$ (M1A1) $\Rightarrow \phi = \int^u f_*(s)ds + g(v)$

11 ↑ BW $\Rightarrow \phi = f(u) + g(v)$ (M1A1)

• $\phi = f(x-ct) + g(x+ct)$
$f(x) + g(x) = \phi(x,0)$ (M1A1) $-cf'(x) + cg'(x) = 0 \Rightarrow f(x) = g(x)$ $[+C]$ (M1A1)
no effect on ϕ

$\therefore f(x) = g(x) = \frac{1}{2}\phi(x,0)$ (M1)

$$\therefore \phi(x,t) = \frac{1}{2}\phi(x-ct,0) + \frac{1}{2}\phi(x+ct,0)$$ where

$$\phi(x-ct,0) = \begin{cases} 0 & (-\infty < x-ct \leqslant -a) \\ a+x-ct & (-a \leqslant x-ct \leqslant 0) \\ a-x+ct & (0 \leqslant x-ct \leqslant a) \\ 0 & (a \leqslant x-ct < \infty) \end{cases}$$ (M2A2)

$$\phi(x+ct,0) = \begin{cases} 0 & (-\infty < x+ct \leqslant -a) \\ a+x+ct & (-a \leqslant x+ct \leqslant 0) \\ a-x-ct & (0 \leqslant x+ct \leqslant a) \\ 0 & (a \leqslant x+ct < \infty) \end{cases}$$

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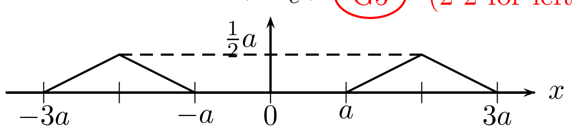
• $ct = 2a$

$\phi(x, \frac{2a}{c})$ (G5) (2-2 for left/right + 1 for height)



5 ↑ SIMILAR EXAMPLES SEEN

(25)

2.

(i) •

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}; \qquad \text{set } u = XT \qquad M1A1$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \qquad M1A1$$

- Take

M1A1 M1A1

$$\frac{X''}{X} = -k^2 \qquad \Rightarrow \qquad X = A\cos kx + B\sin kx$$

- From BCs:

$$M1A1 \quad \begin{cases} u = 0 & \text{at } x = 0 \quad \Rightarrow \quad A = 0, \\ u = 0 & \text{at } x = a \quad \Rightarrow \quad \sin ka = 0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}, n \in \mathbb{N} \end{cases}$$

- Now

M1A1 M1A1

$$\frac{T''}{T} = -c^2 k = -\left(\frac{cn\pi}{a}\right)^2 \Rightarrow T = C\cos\frac{n\pi ct}{a} + D\sin\frac{n\pi ct}{a}$$

- From IC:

$$\frac{\partial u}{\partial t} = 0 \text{ at } t = 0 \qquad \forall x \Rightarrow D = 0. \qquad M1A1$$

- $\Rightarrow$ GS

$$\therefore u(x,t) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{a}\cos\frac{cn\pi t}{a}$$

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(ii)

$$u = f(x) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi t}{a} \qquad \text{at } t = 0 \qquad M2A1$$

$$\Rightarrow A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad M1A2$$

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(iii)

$$\sin\frac{n\pi x}{a}\cos\frac{n\pi ct}{a} = \frac{1}{2}\sin\frac{n\pi}{a}(x+ct) + \frac{1}{2}\sin\frac{n\pi}{a}(x-ct) \qquad M1A1$$

Waves move LEFT / RIGHT at speed c. A1

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3. Substituting the given form of ϕ into the wave equation gives

$$X_1''X_2X_3 + X_1X_2''X_3 + X_1X_2X_3'' = -\frac{\omega^2}{c^2}X_1X_2X_3$$ M1A1

$$\div X_1X_2X_3 \Rightarrow \frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} = -\frac{\omega^2}{c^2} \quad (*)$$ M1A1

Now $\frac{X_1''}{X_1}, \frac{X_2''}{X_2}, \frac{X_3''}{X_3}$ must each be a <u>constant</u>; otherwise $(*)$ could be violated by varying x_1 if $\frac{X_1''}{X_1}$ is not constant etc. M2

• Suppose $\frac{X_1''}{X_1} = k_1^2 \;\Rightarrow x_1 = A\cosh k_1x_1 + B\sinh k_1x_1$

$X_1' = 0$ at $x_1 = 0 \Rightarrow B = 0 \Rightarrow X_1 = A\cosh k_1x_1$ and $X_1' = 0$ at $x_1 = a_1$

$\Rightarrow k_1A\sinh k_1a_1 = 0 \Rightarrow A = 0.$ UNHELPFUL M1A1

• Suppose $\frac{X_1''}{X_1} = 0 \;\Rightarrow X_1 = Ax_1 + B$

$X_1' = 0$ at $x_1 = 0 \Rightarrow A = 0 \Rightarrow X_1 = B$

This satisfies $X_1' = 0$ at $x_1 = a_1$ ACCEPTABLE M1A1

• Suppose $\frac{X_1''}{X_1} = -k_1^2 \;\Rightarrow X_1 = A\cos k_1x_1 + B\sin k_1x_1$

$X_1' = 0$ at $x_1 = 0 \Rightarrow B = 0 \Rightarrow X_1 = A\cos k_1x_1$ and $X_1' = 0$ at $x_1 = a_1$

$\Rightarrow k_1A\sin k_1a_1 = 0 \Rightarrow k_1a_1 = n_1\pi$ where n_1 is a positive integer. M1A1

• Thus $X_1 = A\cos\frac{n_1\pi x_1}{a_1}$ where n_1 is a positive integer or zero. M1A1

We must have $X_2 \propto \cos\frac{n_2\pi x_2}{a_2},\; X_3 \propto \cos\frac{n_3\pi x_3}{a_3}$, and (n_1, n_2, n_3) must (A1)

satisfy $\left(\frac{n_1\pi}{a_1}\right)^2 + \left(\frac{n_2\pi}{a_2}\right)^2 + \left(\frac{n_3\pi}{a_3}\right)^2 = \frac{\omega^2}{c^2} \Rightarrow \frac{n_1^2}{A_1^2} + \frac{n_2^2}{A_2^2} + \frac{n_3^2}{A_3^2} = 1$ → M1A2

18 [NB At least one of n_1, n_2, n_3 must be non-zero.]

Consider the number of triples (n_1, n_2, n_3) within the given surface.

Each lies in the "positive" octant. B1

Surround each triple (n_1, n_2, n_3) by a cube of side 1. M2A1

The volume is therefore approximately the number of such cubes $\times 8$, i.e. the number of

7 different waves is $\frac{1}{8}\cdot\frac{4\pi}{3}A_1A_2A_3 = \frac{\omega^3}{6\pi^2c^3}a_1a_2a_3$ → M2A1

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4.

(i) (a)

Irrotational flow $\Rightarrow$ $\underline{u} = \nabla\phi$ & mass conservation for incompressible flow $\Rightarrow$ $\nabla \cdot \underline{u} = \boxed{\nabla^2\phi = 0.}$ B2

(b)

Bottom does not allow fluid in $\Rightarrow$ $\underline{n} \cdot \nabla\phi = 0$ at $z = -h \Rightarrow$

$$\boxed{\frac{\partial\phi}{\partial z} = 0 \text{ at } z = -h}$$ B2

(c)

The fluid velocity's vertical component at $z = \eta$ must be equal to $\dfrac{\partial\eta}{\partial t}$.

Linear theory $\Rightarrow$ apply at $z = 0 \Rightarrow$ $\boxed{\left.\dfrac{\partial\phi}{\partial z}\right|_{z=0} = \dfrac{\partial\eta}{\partial t}}$ B2

(d)

Bernoulli's integral at $z = \eta \Rightarrow \dfrac{\partial\phi}{\partial t} + g\eta = 0$ at $z = \eta$ since $\dfrac{1}{2}\underline{u}^2$ is negligible and pressure is constant. As in (c), can apply at $z = 0 \Rightarrow$

$$\boxed{\left.\frac{\partial\phi}{\partial t}\right|_{z=0} = -g\eta.}$$ B2

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(ii) $\Phi = f(z)\sin kx \sin\omega t, \quad \eta = a\sin kx\cos\omega t$

(a)

M1A1

$$\Rightarrow f'' = k^2 f \quad \Rightarrow \quad f = A\cosh k(z+h) + B\sinh k(z+h).$$

(b)

$\Rightarrow B = 0 \quad \Rightarrow \quad \boxed{f(z) = A\cosh k(z+h).}$ M1A1

(c)

M1A1

$\Rightarrow kA\sinh kh = -a\omega;$ (d) $\Rightarrow \omega A\cosh kh = -ga$ M1A1

$$\Rightarrow \frac{k}{\omega}\tanh kh = \frac{\omega}{g} \quad \Rightarrow \quad \boxed{\omega^2 = gk\tanh kh} \quad \& \quad \boxed{A = -\frac{a\omega}{k\sinh kh}}$$

M1A1 A1

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(iii)

$$c = \frac{\omega}{k} = \sqrt{\frac{g\tanh kh}{k}}$$ B1

$$c_g = \frac{d\omega}{dk} \quad \Rightarrow \quad 2\omega c_g = g\tanh kh + gkh\,\text{sech}^2 kh$$ M1A1

$$\Rightarrow c_g = \frac{1}{2}\left(\frac{g\tanh kh}{k}\right)^{1/2}\left[1 + \frac{2kh}{\sinh 2kh}\right]$$ M1A1

$$\boxed{\frac{c_g}{c} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right]}$$ M1

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5.

(i)

$$c_t = \frac{dc}{d\rho}\rho_t, \qquad c_x = \frac{dc}{d\rho}\rho_x$$ *M2 + A1 + A1*

$$c_t + cc_x = \frac{dc}{d\rho}\left[\rho_t + c\rho_x\right] = 0$$ *M1A1*

From associated equation

$$\frac{dt}{1} = \frac{dx}{c} = \frac{dc}{0} \Rightarrow c = \text{ const. on curves } \frac{dx}{dt} = c$$ *M1A1*

But c is const. on these curves $\Rightarrow \frac{dx}{dt} =$ const. $\Rightarrow$ curves are straight lines. *M1A1*

Consider the curve through the point with $x = \xi$ at $t = 0$. Then on this curve

$c = \text{const.} = c\{\rho(x,0)\} = c\{f(\xi)\} = F(\xi)$ and equation of curve is $\boxed{x = \xi + F(\xi)t.}$

M1A1 + A1

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(ii) (a)

Suppose $q(\rho) = \alpha\rho^2 + \beta\rho + \gamma$ *M1A1*

M1A1 *A1*

Then $U = \dfrac{\alpha\rho_1^2 + \beta\rho_1 - \alpha\rho_2^2 - \beta\rho_2}{\rho_1 - \rho_2} = \alpha(\rho_1 + \rho_2) + \beta$

But $\frac{1}{2}(c_1 + c_2) = \frac{1}{2}(2\alpha\rho_1 + \beta + 2\alpha\rho_2 + \beta)$ since $q'(\rho) = c(\rho)$.

$\frac{1}{2}(c_1 + c_2) = \alpha(\rho_1 + \rho_2) + \beta = U.$ *M1A1*

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(b)

Under the conditions, for a weak shock, we can approximate $q(\rho)$ by the first three terms of its Taylor series, i.e. approximate $q(\rho)$ by $q(\rho) \approx q(\rho_1) + (\rho - \rho_1)q'(\rho_1) + \frac{1}{2}(\rho - \rho_1)^2 q''(\rho_1)$

(or $\rho_1 \to \rho_2$) *M2A1*

Thus $q(\rho)$ is approx. a quadratic function of $\rho \Rightarrow$ (a) holds. *M1A1*

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