1 *FUNDAMENTALS*

1.1 Vectors and products of vectors

- Vectors are usually in **bold** in printed text, e.g. the **position vector** $\boldsymbol{r} = (x, y, z)$, and with an underline for handwritten text, $\underline{r}$. The vector from point O to a point P is sometimes written $\overrightarrow{OP}$.

- The **magnitude** of the position vector is $r = |\boldsymbol{r}| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, and the unit vector in the **direction** of $\boldsymbol{r}$ is indicated by the 'hat', $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. (The underline would be important here!)

- There are many ways to spell out components:

$$\begin{aligned} \boldsymbol{a} \;=\; (a_x, a_y, a_z) \;=\; \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} &= [a_x\ a_y\ a_z]^T \;=\; a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} \\ &= a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} + a_z\hat{\boldsymbol{z}} \;=\; a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3\,. \end{aligned} \tag{1.1}$$

  Note equivalence of the unit vectors, e.g. $\boldsymbol{k} = \hat{\boldsymbol{z}} = \boldsymbol{e}_3$.

- The **scalar product** or **dot product** of two vectors is given by

$$\boldsymbol{a}\cdot\boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta\,, \tag{1.2}$$

  where θ is the angle between the two vectors.

- The **vector product** or **cross product** is given by

$$\begin{aligned} \boldsymbol{a}\times\boldsymbol{b} \;=\; \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} &= (a_y b_z - a_z b_y)\boldsymbol{i} - (a_x b_z - a_z b_x)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k}\,, \\ &= \hat{\boldsymbol{n}}\,|\boldsymbol{a}|\,|\boldsymbol{b}|\sin\theta\,, \end{aligned} \tag{1.3}$$

  where $\hat{\boldsymbol{n}}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

1.2 Derivatives, small changes and the gradient operator

- Note the difference between d, δ and ∂:
  d is reserved for total derivatives. ∂ denotes the partial derivative. A small but non-zero change in a variable is denoted by δ, e.g., if $f = f(x)$, then for small δx,

$$\delta f \approx \frac{\mathrm{d}f}{\mathrm{d}x}\,\delta x\,.$$

- The following are equivalent notations for the partial derivative:

$$\frac{\partial f}{\partial x}, \quad \partial_x f, \quad f_x.$$

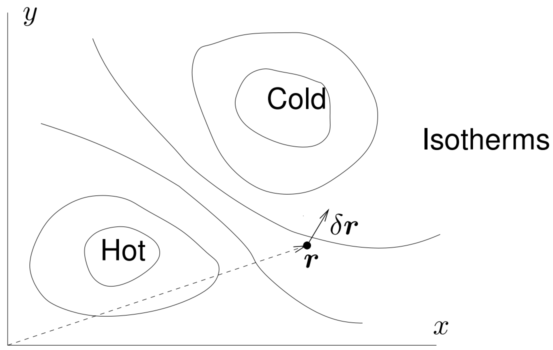
  Avoid the form f_x — it could be confused with $\boldsymbol{f} = (f_x, f_y, f_z)$.

- For a scalar function of position, $\phi = \phi(x, y, z)$, we define

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}, \qquad \text{i.e.} \quad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{1.4}$$

  $\nabla = (\partial_x, \partial_y, \partial_z)$ is the **gradient operator**, often called '**grad**', sometimes 'del' or 'nabla'. It is a **vector operator**, and $\nabla\phi$ is the 'vector gradient' of ϕ.

- Suppose $\phi = \phi(\boldsymbol{r}) = \phi(x, y, z)$ is the (scalar) temperature at position $\boldsymbol{r} = (x, y, z)$.

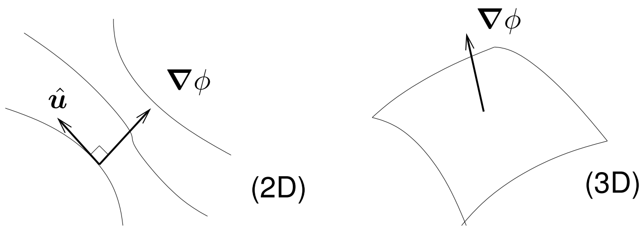


  In 2D, $\phi = const$ defines a contour (or isocontour, or level curve). In 3D, $\phi = const$ defines a **surface** (or isosurface).
  For a small change in position, $\boldsymbol{\delta r} = (\delta x, \delta y, \delta z)$, we observe a small change $\delta\phi$ in temperature. By the first-order Taylor expansion ('Taylor's approximation for small increments'), the change is given by

$$\delta\phi \approx \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z = \boldsymbol{\delta r}\cdot\nabla\phi. \tag{1.5}$$

  If $\hat{\boldsymbol{u}}$ is a unit vector in the direction $\boldsymbol{\delta r}$, then $\hat{\boldsymbol{u}}\cdot\nabla\phi$ gives the rate of change of ϕ in the direction of $\boldsymbol{\delta r}$. We call $\hat{\boldsymbol{u}}\cdot\nabla$ the **directional derivative**.

- If $\hat{\boldsymbol{u}}\cdot\nabla\phi = \delta\phi = 0$, then $\phi = const$. Also, the property of the dot product implies that $\hat{\boldsymbol{u}}$ is perpendicular to $\nabla\phi$. Therefore, $\nabla\phi$ is a vector perpendicular to isocontours in 2D, and **normal to isosurfaces** in 3D.



- Note that $\delta\phi = \hat{\boldsymbol{u}}\cdot\nabla\phi$ is maximised when $\hat{\boldsymbol{u}}$ points in the same direction as $\nabla\phi$. i.e. $\nabla\phi$ gives the **direction of most rapid increase** in ϕ. The rate of change of ϕ in this direction is $|\nabla\phi|$.

- The gradient of a vector is not defined (in this course).

Worked Example 1.2
(A) $\phi = 2x - 3y + z$. Show that $\nabla\phi$ is a constant vector.
(B) Consider $\phi = r^n$. Find $\partial_x r$ and ∇r^n. [Useful result: $\partial r/\partial x = x/r$]

1.3 Motion of a point mass

- Almost always, begin with **Newton's second law**: $\boldsymbol{F} = m\boldsymbol{a}$.
  Really, this follows from the statement that force equals the rate of change of linear momentum $\boldsymbol{p} = m\boldsymbol{v}$. Assuming a constant mass, we have

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a}\,. \tag{1.6}$$

- If several forces act on a point body, they are added and the total net force used in Newton's second law. If the net force on a particle is zero, then its velocity remains constant (**Newton's first law**).

- **Newton's third law** is often stated as 'every action has an equal and opposite reaction'. In other words, if body 1 exerts a force on body 2, $\boldsymbol{F}_{1\to 2}$, then body 2 will exert an equal and opposite force on body 1, $\boldsymbol{F}_{2\to 1} = -\boldsymbol{F}_{1\to 2}$. This is related to the principle of **conservation of linear momentum**:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}\boldsymbol{p}_1}{\mathrm{d}t} = -\frac{\mathrm{d}\boldsymbol{p}_2}{\mathrm{d}t} \quad \Rightarrow \quad \boldsymbol{F}_{2\to 1} = -\boldsymbol{F}_{1\to 2} \tag{1.7}$$

- The **kinetic energy** of a particle is denoted by E, sometimes E_K or T, and is given by

$$E = \frac{1}{2}m\boldsymbol{v}^2 = \frac{\boldsymbol{p}^2}{2m}. \tag{1.8}$$

  where $\boldsymbol{v}^2 = |\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v}$.
  If present, a **potential energy** is denoted by V or E_P, and its form depends on the situation under consideration. For example, close to the surface of the Earth one might use $V = mgh$ with $g = +9.81\,m/s^2$. (The origin of this potential will be covered in Chapter 2.)
  The principle of **conservation of energy** states that the total energy does not change over time, e.g. $E + V = const$.

Worked Example 1.3 (Free-fall of an object from rest)
Calculate the speed of an object in free-fall after a falling a distance h, where it is released from rest at time $t = 0$, using
(A) Newton's second law,
(B) conservation of energy, with the potential $V = mgh$.

2 *THE WORK-ENERGY PRINCIPLE*

2.1 The work-energy principle for a particle moving under a force

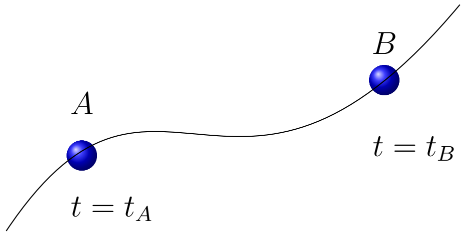
The **kinetic energy (KE)** of a particle is given by

$$E = \frac{1}{2}m\,v^2 = \frac{1}{2}m\,\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\,, \tag{2.1}$$

where $\dot{\mathbf{r}} = \mathbf{d}\mathbf{r}/dt$, $\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$, etc. The rate of change of E is then

$$\dot{E} = m\,\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = (m\,\boldsymbol{a})\cdot\dot{\mathbf{r}} = \mathbf{F}\cdot\dot{\mathbf{r}}\,, \tag{2.2}$$

using Newton's second law.
Consider the line integral between two points on the particle's path.



The increase in KE from point A to point B is given by

$$\int_{t_A}^{t_B} \dot{E}\,\mathrm{d}t = \Big[E\Big]_{t_A}^{t_B} = E_B - E_A$$

Thus

$$E_B - E_A = \int_{t_A}^{t_B} \mathbf{F}\cdot\dot{\mathbf{r}}\;\mathrm{d}t \tag{2.3}$$

At the same time, from the principle that 'work done = force $\times$ distance', as the particle moves from $\mathbf{r}$ to $\mathbf{r} + \boldsymbol{\delta r}$ in time δt, we have that

$$\delta W = \mathbf{F}\cdot\boldsymbol{\delta}\mathbf{r} = \mathbf{F}\cdot(\dot{\mathbf{r}}\,\delta t)\,.$$

Therefore, the total work done by $\mathbf{F}$ in going from A to B is

$$W = \int_A^B \mathbf{F}\cdot\mathbf{dr} = \int_{t_A}^{t_B} \mathbf{F}\cdot\dot{\mathbf{r}}\;\mathrm{d}t\,. \tag{2.4}$$

Together, (2.3) and (2.4) give the **work-energy principle**:

$$\boxed{E_B - E_A = W} \tag{2.5}$$

i.e. the change in KE of the particle is equal to the total work done by the force on the particle.
The time derivative of (2.4) implies that

$$\dot{W} = \mathbf{F}\cdot\dot{\mathbf{r}} = \dot{E},$$

i.e. the rate of change of KE is equal to the rate of work done, or **power**, imparted by the force.

Example 2.1.1 (Free-fall of an object from rest)
In Worked Example 1.3 it was shown that the potential, $V = mgh$, where h is the height fallen, gives the same velocity (and Kinetic energy) as application of Newton's laws.
This motion is linear. Observe that the work done during the fall is 'force $\times$ distance' $= W = V = mg \times h$.

Worked Example 2.1 (Projectile under gravity)
Verify (2.5) for a projectile moving under gravity, with no air resistance, where it is sent from point A at a speed U and angle α to the horizontal, and the projectile is at point B at time $t_B = \tau$.

2.2 Line Integrals and Conservative Forces

The integral (2.4) along the curve C connecting points A and B,

$$W = \int_C \mathbf{F} \cdot \mathbf{dr}\,, \tag{2.6}$$

is a **line integral**. The work done, W, is the integral of the scalar product of the force and the displacement vector along the line. In general, the value of a line integral depends on the path taken between A and B.
If there exists a $V = V(\mathbf{r})$ such that

$$\mathbf{F} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right), \qquad \text{i.e.} \quad \boxed{\mathbf{F} = -\boldsymbol{\nabla} V}, \tag{2.7}$$

then $\mathbf{F}$ is called **conservative**, and W does **not** depend on the path C taken from point A to B. When it exists, V is called the **potential energy** (PE).

Proof [W independent of path from A to B if $\mathbf{F} = -\boldsymbol{\nabla} V$]
From (2.4),

$$W = \int_{t_A}^{t_B} \mathbf{F} \cdot \dot{\mathbf{r}}\,\mathrm{d}t = -\int_{t_A}^{t_B} \boldsymbol{\nabla} V \cdot \dot{\mathbf{r}}\,\mathrm{d}t\,. \tag{2.8}$$

Using the Taylor expansion of $V = V(\mathbf{r})$ over a small change in position $\boldsymbol{\delta}\mathbf{r} = (\delta x,\, \delta y,\, \delta z)$

$$\delta V = \frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z \quad \Rightarrow \quad \left.\frac{\partial V}{\partial t}\right|_{\mathbf{r}(t)} = \boldsymbol{\nabla} V \cdot \dot{\mathbf{r}}\,.$$

Thus

$$W = -\int_{t_A}^{t_B} \frac{\partial V}{\partial t}\,\mathrm{d}t = [V]_{t_A}^{t_B} = V_A - V_B\,. \tag{2.9}$$

Therefore the result is dependent on the positions A and B, but is independent of the path taken from A to B. ■

Where V exists, with the work-energy principle (2.5), we have that

$$W = V_A - V_B = E_B - E_A \quad \Rightarrow \quad E_A + V_A = E_B + V_B\,.$$

As the points A and B are arbitrary, it follows that everywhere

$$\boxed{E + V = \text{constant}} \tag{2.10}$$

i.e. the **principle of conservation of energy**.

Example 2.2.1 (The gravitational potential, valid near the surface of Earth)
Here $V = mgh$ (see e.g. Example 2.1.1). Identifying height h with the dimension z, we have

$$\mathbf{F} \;=\; -\boldsymbol{\nabla} V \;=\; -\boldsymbol{\nabla}(mgz) \;=\; -\partial_z(mgz)\,\mathbf{k} = -mg\,\mathbf{k}\,,$$

i.e. a downwards force of magnitude mg.

Recall (section 1.2) that $V = const$ defines a surface and $\boldsymbol{\nabla} V$ defines a normal to the surface, pointing in the direction of maximum rate of increase of V with respect to change in position. Here $V = const \;\Rightarrow\; z = const$ is a surface of constant gravitational potential energy (GPE), and $\boldsymbol{\nabla} V$ points directly 'up', in the $\mathbf{k}$ direction.

Worked Example 2.2 (Gravitational field of a spherical body)
(A) Evaluate the force $\mathbf{F}$ for the potential

$$V \;=\; V(r) \;=\; -\frac{GMm}{r}\,, \qquad G \approx 6.67 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\text{s}^{-2},$$

where M, m are constant masses and G is the universal gravitational constant.

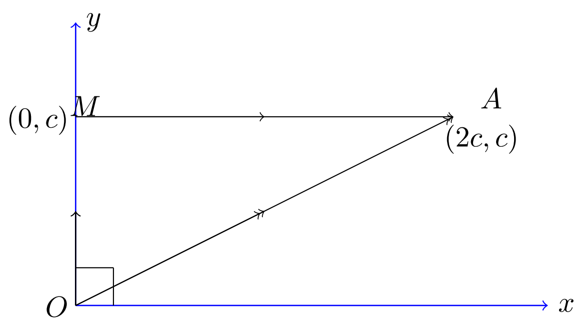
Example 2.2.2 (Line integral for a non-conservative force)
Given that

$$\mathbf{F} = \frac{P}{c}(3y\,\mathbf{i} + x\,\mathbf{j})$$

where P and c are constants, evaluate

$$W = \int_0^A \mathbf{F}\cdot\mathbf{dr}$$

along
Path 1: $OM \;+\; MA$ and
Path 2: OA.



$$\boldsymbol{\delta}\mathbf{r} = \delta x\;\mathbf{i} + \delta y\;\mathbf{j} + \delta z\;\mathbf{k},$$

$$\delta W \;=\; \mathbf{F}\cdot\boldsymbol{\delta}\mathbf{r} = \frac{P}{c}(3y\,\delta x + x\,\delta y).$$

Path 1:
<u>On OM</u>: $x = 0$ and $\delta x = 0 \;\;\Rightarrow\;\; \delta W = 0 \;\;\Rightarrow\;\; \int_{OM} dW = 0.$
<u>On MA</u>: $y = c$ and $\delta y = 0 \;\;\Rightarrow\;\; \delta W = 3P\,\delta x \;\;\Rightarrow\;\; \int_{MA} dW = 3P\int_0^{2c} dx = 6Pc$.

$$\Rightarrow \quad W_1 = 0 + 6\,P\,c = 6\,P\,c.$$

Path 2:
<u>On OA</u>: $y = \frac{1}{2}x \;\Rightarrow\; \delta y = \frac{1}{2}\delta x \quad\Rightarrow\quad \delta W = \frac{P}{c}(\frac{3}{2}x\,\delta x + x\,\frac{1}{2}\delta x) = \frac{2P}{c}x\,\delta x$

$$\Rightarrow \quad \int_{OA} dW = \frac{2P}{c}\int_0^{2c} x\,dx = 4Pc \quad\Rightarrow\quad W_2 = 4\,P\,c.$$

The result depends on the path taken, and therefore $\mathbf{F}$ is **not** conservative. We therefore do **not** expect to be able to find a V such that $\mathbf{F} = -\boldsymbol{\nabla} V$.
Suppose there were such a V, i.e.

$$\begin{aligned}
\text{(i)} \quad -\frac{\partial V}{\partial x} &= \frac{3P}{c}y, \\
\text{(ii)} \quad -\frac{\partial V}{\partial y} &= \frac{P}{c}x, \\
\text{(iii)} \quad -\frac{\partial V}{\partial z} &= 0.
\end{aligned}$$

(i) and (iii) imply that

$$V = -\frac{3P}{c}xy + f(y)\,,$$

where $f(y)$ is an arbitrary function of y. But then (ii) gives

$$-\frac{\partial V}{\partial y} = \frac{3P}{c}x - f'(y) \neq \frac{Px}{c}\,.$$

Equality is not possible for any choice of $f(y)$. Therefore there is no potential energy V which satisfies (i)–(iii).

Worked Example 2.2 (Line integral for a conservative force)
(B) Calculate the work done along the same paths as in Example 2.2.2, but now with the force

$$\mathbf{F} = \frac{P}{c}(3x\,\mathbf{i} + y\,\mathbf{j}).$$

Show that a potential energy V exists for this case.

Worked Example 2.2 (Escape velocity of a projectile)
(C) A projectile of mass m is fired upwards from the Earth's surface with speed u. Let

$$U_e = \left(\frac{2GM}{a}\right)^{1/2},$$

where M is the mass of the Earth, and a is the radius of the Earth.
If atmospheric resistance can be neglected, show that:
(i) if $u > U_e$ the projectile escapes to infinity,
(ii) if $u < U_e$ the projectile returns to Earth after reaching a maximum distance from the centre of the Earth of $\frac{a}{1-\left(\frac{u}{U_e}\right)^2}$.