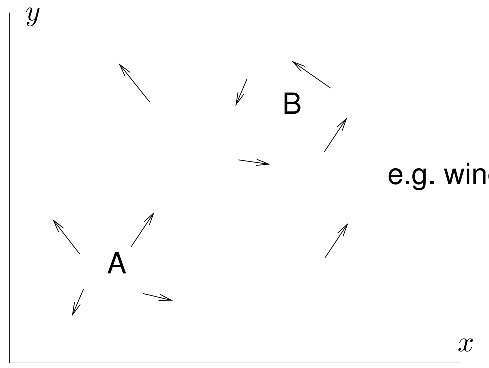
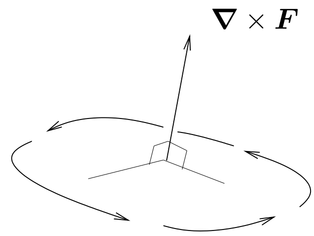
# 3 *VECTOR CALCULUS I*

## 3.1 Divergence and curl of vector fields

Let $\boldsymbol{\nabla} = (\partial_x, \partial_y, \partial_z)$. Several properties of $\boldsymbol{\nabla}\phi$ for scalar $\phi$ were introduced in section 1.2. The gradient operator may also be applied to vector fields. Let $\boldsymbol{F} = (F_1, F_2, F_3) = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$, where $F_1 = F_1(\boldsymbol{r}) = F_1(x, y, z)$ etc.




e.g. wind velocity




The **divergence is a scalar** measuring **net flux** of the field from each point.

$$\operatorname{div}(\boldsymbol{F}) = \boldsymbol{\nabla} \cdot \boldsymbol{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3 \tag{3.1}$$

For point A of the figure, $\boldsymbol{\nabla} \cdot \boldsymbol{F} > 0$. For converging vectors, $\boldsymbol{\nabla} \cdot \boldsymbol{F} < 0$. Near B, $\boldsymbol{\nabla} \cdot \boldsymbol{F} \approx 0$.

The **curl is a vector** giving the **magnitude and axis of rotation** about each point.

$$\operatorname{curl}(\boldsymbol{F}) = \boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix} = (\partial_y F_3 - \partial_z F_2,\ \partial_z F_1 - \partial_x F_3,\ \partial_x F_2 - \partial_y F_1) \tag{3.2}$$

The rotation near B defines a vector $\boldsymbol{\nabla} \times \boldsymbol{F}$ pointing out of the page. The vector would point into the page for rotation in the opposite direction. Near A, $\boldsymbol{\nabla} \times \boldsymbol{F} \approx \boldsymbol{0}$.

NOTE: Remember that $\boldsymbol{\nabla}$ is a derivative operator. Just like $\partial_x f \neq f\partial_x$, also $\boldsymbol{\nabla} \cdot \boldsymbol{F} \neq \boldsymbol{F} \cdot \boldsymbol{\nabla}$. $\partial_x f$ and $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ are scalars, whereas $f\partial_x$ and $\boldsymbol{F} \cdot \boldsymbol{\nabla}$ are operators. The latter is similar to the directional derivative of section 1.2.

> **Worked Example 3.1**
> (A) Consider $\boldsymbol{F} = \boldsymbol{r}$. Show that $\boldsymbol{\nabla} \cdot \boldsymbol{r} = 3$ and $\boldsymbol{\nabla} \times \boldsymbol{r} = \boldsymbol{0}$.
> (B) Let $\boldsymbol{F} = x\,\boldsymbol{j}$. Show that $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{k}$.

## 3.2 Important identities

For any scalar field $\phi$,

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{0}\,.} \tag{3.3}$$

and for any vector field $\boldsymbol{F}$,

$$\boxed{\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0\,.} \tag{3.4}$$

Further, it can be shown that for a volume $V$,

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0} \ \text{ in } \ V \quad \Longleftrightarrow \quad \exists\, \psi \ \text{ in } \ V \ \text{ s.t. } \ \boldsymbol{F} = \boldsymbol{\nabla}\psi\,, \tag{3.5}$$

and

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \ \text{ in } \ V \quad \Longleftrightarrow \quad \exists\, \boldsymbol{A} \ \text{ in } \ V \ \text{ s.t. } \ \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}\,. \tag{3.6}$$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$, then $\boldsymbol{F}$ is said to be '**irrotational**' and may be defined in terms of a scalar potential $\psi$. If $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, then $\boldsymbol{B}$ is said to be '**solenoidal**' or '**divergence-free**', and may be defined via a vector potential $\boldsymbol{A}$.

**Example 3.2.1 (Proof of (3.3))**
By definition, $\boldsymbol{\nabla}\phi = (\partial_x\phi,\, \partial_y\phi,\, \partial_z\phi)$. Applying the curl (3.2),

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = (\partial_y\partial_z\phi - \partial_z\partial_y\phi,\, \ldots,\, \ldots)\,.$$

But

$$\partial_y\partial_z \equiv \partial_{yz} \equiv \frac{\partial^2}{\partial y\,\partial z} \equiv \frac{\partial^2}{\partial z\,\partial y} \quad \Rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}\,.$$

## 3.3 The Laplacian operator

The Laplacian operator is defined

$$\nabla^2\phi \equiv \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\phi) = (\partial_{xx} + \partial_{yy} + \partial_{zz})\,\phi\,. \tag{3.7}$$

The **Laplace equation**,

$$\nabla^2\phi = 0\,. \tag{3.8}$$

frequently arises in applications. The Laplacian may be applied to a vector. In Cartesian coordinates we have that

$$\boldsymbol{\nabla}^2\boldsymbol{F} \;=\; (\nabla^2 F_1,\, \nabla^2 F_2,\, \nabla^2 F_3) \;=\; (\nabla^2 F_1)\,\boldsymbol{i} + (\nabla^2 F_2)\,\boldsymbol{j} + (\nabla^2 F_3)\,\boldsymbol{k}\,. \tag{3.9}$$

**Example 3.3.1**
Verify that $\phi = \ln(x^2 + y^2)$ satisfies the Laplace equation.

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$$\partial_x\phi = \frac{1}{x^2 + y^2} \cdot 2x\,,$$

$$\partial_{xx}\phi = \frac{1}{x^2 + y^2} \cdot 2 + \frac{-2x}{(x^2 + y^2)^2} \cdot 2x = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}\,.$$

By symmetry $\partial_{yy}\phi = \dfrac{2(x^2 - y^2)}{(x^2 + y^2)^2} \quad \Rightarrow \quad \partial_{yy}\phi = -\partial_{xx}\phi\,.$

Also $\partial_z\phi = 0$. Together,

$$\nabla^2\phi = \partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0\,.$$

**Worked Example 3.3**
Show that $\nabla^2 f = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\frac{\mathrm{d}}{\mathrm{d}r}f)$ if $f = f(r)$. Hence find the most general $f = f(r)$ s.t. $\nabla^2 f = 0$.

## 3.4 Other important operators

The construction $\nabla \times (\nabla \times \boldsymbol{F})$ is sometimes called the **'double curl'**, and in Cartesian coordinates it can readily be shown that

$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}\,. \tag{3.10}$$

The **advective derivative**, often just called 'v dot grad', is given by

$$\boldsymbol{v} \cdot \nabla = v_1\, \partial_x + v_2\, \partial_y + v_3\, \partial_z\,. \tag{3.11}$$

It is similar to the directional derivative, $\hat{\boldsymbol{v}} \cdot \nabla$ (see section 1.2), and, like the Laplacian, can be applied to both scalar and vector fields:

$$\begin{aligned}
(\boldsymbol{v} \cdot \nabla)\, \phi &= (v_1\, \partial_x + v_2\, \partial_y + v_3\, \partial_z)\, \phi\,,\\
(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{u} &= [(\boldsymbol{v} \cdot \nabla)\, u_1]\, \boldsymbol{i} + [(\boldsymbol{v} \cdot \nabla)\, u_2]\, \boldsymbol{j} + [(\boldsymbol{v} \cdot \nabla)\, u_3]\, \boldsymbol{k}\,.
\end{aligned}$$

**Worked Example 3.4**
Find $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ for the case $\boldsymbol{u} = (U/L)(x, -y, 0)$, where $U$ and $L$ are constants.

## 3.5 Suffix notation and the summation convention

### *3.5.1 Definitions*

The **Kronecker delta**, $\delta_{ij}$ is defined for $i = 1, 2, 3$ and $j = 1, 2, 3$:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The **alternating symbol**, $\epsilon_{ijk}$, is defined

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1),\ (3,1,2) \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (2,1,3),\ (1,3,2) \\ \ \ 0 & \text{for any other } (i,j,k) \text{ (when 2 or more indices equal)} \end{cases}$$

For example, $\delta_{32} = 0,\ \delta_{22} = 1,\ \epsilon_{132} = -1,\ \epsilon_{221} = 0,\ \epsilon_{iik} = 0$. Observe that switching the order of indices on $\epsilon$ reverses the sign, e.g. $\epsilon_{jki} = -\epsilon_{ikj}$ .

### *3.5.2 The summation convention*

- Let $x_1 = x,\ x_2 = y,\ x_3 = z$, so that $\boldsymbol{r}_i = x_i$. Similarly, let $\boldsymbol{e}_1 = \boldsymbol{i},\ \boldsymbol{e}_2 = \boldsymbol{j},\ \boldsymbol{e}_3 = \boldsymbol{k}\,.$
- When an index, e.g. $i$, **occurs only once** in a term of an equation it is called '**a free suffix**', and it **represents all possibilities** from $1$ to $3$.
  **Examples**:
  $x_i = 5 \quad\Rightarrow\quad x_1 = x_2 = x_3 = 5.$
  $a_{ij} = \delta_{ij}\, p \quad\Rightarrow\quad a_{ij} = 0$ for $i \neq j$ and $a_{ij} = p$ for $i = j$.
- When an index **occurs twice** in a term of an equation, the term is **summed** from $1$ to $3$.
  **Example**: $a_k\, b_k = a_1 b_1 + a_2 b_2 + a_3 b_3 = \boldsymbol{a} \cdot \boldsymbol{b}$.

**Example 3.5.1 (Important summation properties)**
(a) $\delta_{jj} = 3$
(b) $\epsilon_{ijk}\epsilon_{ijk} = 6$
(c) $\boxed{\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i}$
(d) $\boldsymbol{r} = x_i \boldsymbol{e}_i$
(e) $\boxed{(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk}\, a_j\, b_k}$
(f) $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c} = a_i\, \epsilon_{ijk}\, b_j\, c_k$

There are a couple of **rules** to check that the notation makes sense! :

- No index may occur more than twice in a single term of an equation.
  E.g. $a_i b_i c_i$ is meaningless.
- If free suffices appear, then they must be the same in every term of the equation.
  E.g. $a_j = c_j$ and $b_{ij} a_i + c_j = 0$ are ok, but $c_j = b_{ij} a_j$ is meaningless.

Using this convention, we may write new forms for the operators of the previous sections.

$$\text{Let } \boxed{\boldsymbol{\nabla}_i = \frac{\partial}{\partial x_i} = \partial_i}\,.$$

$$(1)\ (\boldsymbol{\nabla}\phi)_i = \frac{\partial \phi}{\partial x_i} \qquad (2)\ \boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_i}{\partial x_i} \qquad (3)\ (\boldsymbol{\nabla} \times \boldsymbol{F})_i = \epsilon_{ijk} \frac{\partial F_k}{\partial x_j}$$

$$(4)\ \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_j\, \partial x_j} \qquad (5)\ (\boldsymbol{\nabla}^2 \boldsymbol{F})_i = \frac{\partial^2 F_i}{\partial x_j\, \partial x_j}$$

$$(6)\ (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \phi = v_j \frac{\partial \phi}{\partial x_j} \qquad (7)\ [(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{u}]_i = v_j \frac{\partial u_i}{\partial x_j}$$

**Worked Example 3.5**
(A) Use the suffix notation to show that $\boldsymbol{\nabla} \times (\phi\, \boldsymbol{v}) = \phi \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{\nabla}\phi \times \boldsymbol{v}$.

### *3.5.3 The substitution property of $\delta_{ij}$*

- Consider the term $\delta_{ij} a_j$, where summation over $j$ is implied. Now, $\delta_{ij}$ is non-zero only for one case, $j = i$. Therefore we may simplify:

$$\boxed{\delta_{ij} a_j = a_i}\,. \tag{3.12}$$

In other words, if a **delta has a summed index**, then the delta may be omitted, and in the rest of the term the summed suffix is **substituted by the other index** of the delta.
**Example**: $\delta_{ip}\, \epsilon_{ijk} = \epsilon_{pjk}$ . ($i$ summed, replace by $p$)

**Worked Example 3.5**
Show that
(B) $\delta_{ip}\, \delta_{jq}\, c_p\, d_q = c_i\, d_j$
(C) $\delta_{ij}\, b_{ij} = b_{ii} = b_{jj}$
(D) $\delta_{ij} \epsilon_{ijk} = \epsilon_{iik} = 0$

### *3.5.4 Some useful results*

$$\boxed{\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}} \tag{3.13}$$

Note the repeated suffix $k$ in the property. The remaining suffices $i, j, l, m$ are free in general, none of them occurs more than once in the *same* term. The double-epsilon usually arises in the presence of two vector-products.

> **Example 3.5.2**
> $\{\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})\}_i = \epsilon_{ijk}\, a_j\, (\boldsymbol{b} \times \boldsymbol{c})_k = \epsilon_{ijk}\, a_j\, \epsilon_{klm}\, b_l\, c_m = \epsilon_{ijk}\, \epsilon_{klm}\, a_j\, b_l\, c_m = (\delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl})\, a_j\, b_l\, c_m$
> $$= \underbrace{\delta_{il}}_{l\to i}\, \underbrace{\delta_{jm}}_{m\to j}\, a_j\, b_l\, c_m - \underbrace{\delta_{im}}_{m\to i}\, \underbrace{\delta_{jl}}_{l\to j}\, a_j\, b_l\, c_m = a_j\, b_i\, c_j - a_j\, b_j\, c_i = (\boldsymbol{a}\cdot\boldsymbol{c})\, b_i - (\boldsymbol{a}\cdot\boldsymbol{b})\, c_i$$
> $\forall\ i = 1, 2$ or $3 \quad \Rightarrow \quad \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}$.

Other useful results include:

$$\boxed{\partial x_i/\partial x_j = \delta_{ij}, \qquad x_i x_i = r^2\,,}$$

$$\epsilon_{ijk}\,\delta_{jk} = 0, \qquad \epsilon_{ijk}\,x_j\,x_k = 0, \qquad \epsilon_{ijk}(a_j\,b_k + a_k\,b_j) = 0. \tag{3.14}$$

For the first two, recall $x_1$=$x$, $x_2$=$y$, $x_3$=$z$. The third result follows from the substitution property. The penultimate result states $(\boldsymbol{r} \times \boldsymbol{r})_i = 0$ and the last that $(\boldsymbol{a} \times \boldsymbol{b})_i + (\boldsymbol{b} \times \boldsymbol{a})_i = 0$.

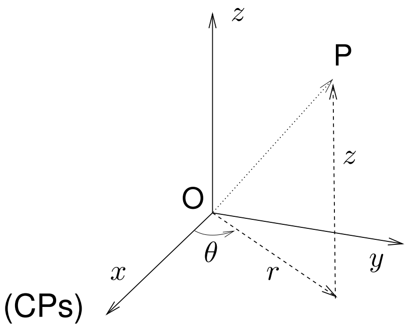
> **Worked Example 3.5**
> (E) Show that $\boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = 2\,\boldsymbol{\Omega}$ for a constant vector $\boldsymbol{\Omega}$.

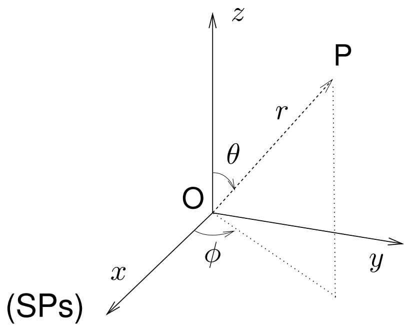
> **Example 3.5.3**
> Find $\nabla^2\phi$ when $\phi = (\boldsymbol{a}\cdot\boldsymbol{r})\, f(r)$, where $\boldsymbol{a}$ is a constant vector.
>
> ---
>
> $$\phi = a_j x_j\, f(r)\,,$$
> $$\frac{\partial\phi}{\partial x_i} = a_j \underbrace{\frac{\partial x_j}{\partial x_i}}_{\delta_{ij}\ j\to i} f + a_j x_j \frac{\mathrm{d}f}{\mathrm{d}r} \underbrace{\frac{\partial r}{\partial x_i}}_{x_i/r} = a_i f + a_j x_j x_i \frac{f'}{r}$$
> $$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i\,\partial x_i} = a_i \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial x_i} + a_j \underbrace{\delta_{ij}}_{j\to i}\, x_i \frac{f'}{r} + a_j x_j \underbrace{\delta_{ii}}_{=3} \frac{f'}{r} + a_j x_j x_i \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f'}{r}\right) \underbrace{\frac{\partial r}{\partial x_i}}_{x_i/r}$$
> On the last term, $x_i x_i = \sum x_i^2 = r^2$. Then
> $$\nabla^2\phi = 5 a_i x_i \frac{f'}{r} + a_j x_j r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{f'}{r}\right) = (\boldsymbol{a}\cdot\boldsymbol{r})\left(f'' + 4\frac{f'}{r}\right)$$

# 4 *VECTOR CALCULUS II*

## 4.1 Cylindrical polar and Spherical polar coordinates

For many problems the Cartesian coordinate system is not the most natural. For example a full sphere is easily described by limits on $r$, as $0 < r < R$, but the sphere is cumbersome to describe in terms of limits on $x$, $y$ and $z$. The following section describes the two most common alternatives to the Cartesian system.
It is important to be familiar with the following diagrams:




For Cylindrical Polars (**CPs**), $(r, \theta, z)$:

$$\begin{aligned} x &= r\cos\theta, & y &= r\sin\theta, & z &= z, \\ r &= (x^2+y^2)^{\frac{1}{2}}, & \theta &= \tan^{-1}\tfrac{y}{x}, & & \\ r &\geq 0, & 0 &\leq \theta \leq 2\pi & & \end{aligned} \tag{4.1}$$

For Spherical Polars (**SPs**), $(r, \theta, \phi)$:

$$\begin{aligned} x &= r\sin\theta\cos\phi, & y &= r\sin\theta\sin\phi, & z &= r\cos\theta, \\ r &= (x^2+y^2+z^2)^{\frac{1}{2}}, & \theta &= \cos^{-1}\left\{\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}}\right\}, & \phi &= \tan^{-1}\tfrac{y}{x} \\ r &\geq 0, & 0 &\leq \theta \leq \pi, & 0 &\leq \phi \leq 2\pi \end{aligned} \tag{4.2}$$

**In SPs, $r$ is the distance from the origin O**, as usual. $\boldsymbol{r} = r\,\hat{\boldsymbol{r}}$ (SPs).
**In CPs, $r$ is the distance from the $z$-axis**, the cylindrical-radius. $\boldsymbol{r} = r\,\hat{\boldsymbol{r}} + z\hat{\boldsymbol{z}}$ (CPs).

Note that $\theta$ (CPs) $\equiv \phi$ (SPs). When confusion might be possible, CPs are sometimes written $(s, \phi, z)$ rather than $(r, \theta, z)$.

Let the coordinates $(\alpha_1,\ \alpha_2,\ \alpha_3)$ denote e.g. $(x, y, z)$ or $(r, \theta, z)$, or one of the many orthogonal coordinate systems. $\hat{\boldsymbol{\alpha}}_1$, $\hat{\boldsymbol{\alpha}}_2$, $\hat{\boldsymbol{\alpha}}_3$ denote **unit vectors**, e.g. CPs $\hat{\boldsymbol{\alpha}}_1 = \hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\alpha}}_2 = \hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\alpha}}_3 = \hat{\boldsymbol{z}}$.

At every point the unit vectors for the coordinates are **orthogonal**, i.e. $\hat{\boldsymbol{\alpha}}_i \cdot \hat{\boldsymbol{\alpha}}_j = 0$ if $i \neq j$, and they are ordered such that $\hat{\boldsymbol{\alpha}}_1 \times \hat{\boldsymbol{\alpha}}_2 = \hat{\boldsymbol{\alpha}}_3$, etc., e.g. in Cartesian coordinates $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$.

**Worked Example 4.1**
(A) Sketch coordinate surfaces, corresponding to $\alpha_i = const$, for CPs and SPs.
(B) Sketch the unit vectors for CPs.

## 4.2 The line element revisited

Let P and P' be neighbouring points, with $\overrightarrow{OP} = \boldsymbol{r}$ and $\overrightarrow{OP'} = \boldsymbol{r} + \boldsymbol{\delta l}$. The notations $\boldsymbol{\delta r}$ and $\boldsymbol{\delta s}$ are also commonplace. So far we've used the former, but if there is potential for ambiguity, when $r$ is a coordinate direction, then $\boldsymbol{\delta l}$ is safer.

In Cartesians we write the small change in position $\boldsymbol{\delta l} = \delta x\, \boldsymbol{i} + \delta y\, \boldsymbol{j} + \delta z\, \boldsymbol{k}$. In general we write,

$$\boldsymbol{\delta l} = \delta l_1 \hat{\boldsymbol{\alpha}}_1 + \delta l_2 \hat{\boldsymbol{\alpha}}_2 + \delta l_3 \hat{\boldsymbol{\alpha}}_3 = h_1\, \delta\alpha_1\, \hat{\boldsymbol{\alpha}}_1 + h_2\, \delta\alpha_2\, \hat{\boldsymbol{\alpha}}_2 + h_3\, \delta\alpha_3\, \hat{\boldsymbol{\alpha}}_3\,. \tag{4.3}$$

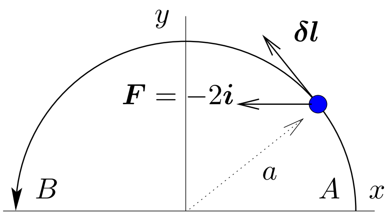
As $\boldsymbol{\delta l}$ is a change in position, each of $\delta l_1 = h_1\, \delta\alpha_1$ etc. must be a length. This defines the **scale factors** $h_i$, and where $\delta\alpha_i$ is an angle $h_i$ must be a length.

For Cartesians we have $h_1 = h_2 = h_3 = 1$. It can be shown that for

$$\begin{aligned} &\text{CPs } (r, \theta, z): \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1, \\ &\text{SPs } (r, \theta, \phi): \quad h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta\,. \end{aligned} \tag{4.4}$$

**Worked Example 4.2**
Check (4.4) via sketches.

**Example 4.2.1 (Line integral on a curved path in CPs)**




Path: $\theta = \alpha_2 \quad \Rightarrow \quad \boldsymbol{\delta l} = h_2\, \delta\alpha_2\, \hat{\boldsymbol{\alpha_2}} = r\, \delta\theta\, \hat{\boldsymbol{\theta}}$, with $r = a\,.$

Then $\displaystyle\int_A^B \boldsymbol{F} \cdot \boldsymbol{\delta l} = -2a \int_0^\pi \boldsymbol{i} \cdot \hat{\boldsymbol{\theta}}\, \mathrm{d}\theta = 2a \int_0^\pi \sin\theta\, \mathrm{d}\theta = 4a\,.$

## 4.3 Generalised gradient operators

**The physical interpretations remain unchanged** for grad, div and curl. The divergence gives the same scalar, grad and curl give the same vector, but the representation will look different in another coordinate system. For curvilinear orthogonal coordinates $(\alpha_1, \alpha_2, \alpha_3)$, not all of the $\alpha_i$ have the dimension of length and the unit vectors $\hat{\boldsymbol{\alpha}}_i$ are not constant. Apart from the generalised gradient, proofs of the following are lengthy.

- **Gradient:**
  The gradient gives the change in the scalar $\phi$ for a change in position $\boldsymbol{\delta l}$. Therefore

$$(\boldsymbol{\nabla}\phi)_1 \approx \frac{\delta\phi}{\delta l_1} = \frac{\delta\phi}{h_1\, \delta\alpha_1} \quad \Rightarrow$$

$$\boxed{\boldsymbol{\nabla}\phi = \left( \frac{1}{h_1} \frac{\partial\phi}{\partial\alpha_1}, \frac{1}{h_2} \frac{\partial\phi}{\partial\alpha_2}, \frac{1}{h_3} \frac{\partial\phi}{\partial\alpha_3} \right)} \tag{4.5}$$

- **Divergence:**

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{h_1\, h_2\, h_3} \left\{ \frac{\partial}{\partial\alpha_1}(h_2\, h_3\, F_1) + \frac{\partial}{\partial\alpha_2}(h_1\, h_3\, F_2) + \frac{\partial}{\partial\alpha_3}(h_1\, h_2\, F_3) \right\} \tag{4.6}$$

- **Curl:**

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{h_1\, h_2\, h_3} \begin{vmatrix} h_1\, \hat{\boldsymbol{\alpha}}_1 & h_2\, \hat{\boldsymbol{\alpha}}_2 & h_3\, \hat{\boldsymbol{\alpha}}_3 \\ \frac{\partial}{\partial\alpha_1} & \frac{\partial}{\partial\alpha_2} & \frac{\partial}{\partial\alpha_3} \\ h_1\, F_1 & h_2\, F_2 & h_3\, F_3 \end{vmatrix} \tag{4.7}$$

- **Scalar Laplacian:**
  By definition, $\nabla^2\phi = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)$. This can be calculated from the above to be

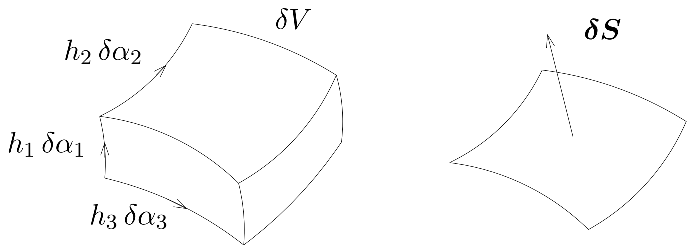
$$\nabla^2\phi = \frac{1}{h_1\,h_2\,h_3}\left\{\frac{\partial}{\partial\alpha_1}\left(\frac{h_2\,h_3}{h_1}\,\frac{\partial\phi}{\partial\alpha_1}\right) + \frac{\partial}{\partial\alpha_2}\left(\frac{h_1\,h_3}{h_2}\,\frac{\partial\phi}{\partial\alpha_2}\right) + \frac{\partial}{\partial\alpha_3}\left(\frac{h_1\,h_2}{h_3}\,\frac{\partial\phi}{\partial\alpha_3}\right)\right\} \tag{4.8}$$

- **Vector Laplacian:**
  The following expansion formulae, which can be shown to hold for Cartesians, we insist holds for all coordinate systems:

$$\boldsymbol{\nabla}^2\mathbf{F} \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F}), \tag{4.9}$$

  The RHS can be calculated from formulae above. The result $\boldsymbol{\nabla}^2\mathbf{F} = (\nabla^2 F_i)\,\hat{\boldsymbol{\alpha}}_i$ holds for Cartesians but *not* in general.

## 4.4 Volume and surface elements




From the previous sections, the line element is given by $\boldsymbol{\delta l} = (h_1\,\delta\alpha_1,\ h_2\,\delta\alpha_2,\ h_3\,\delta\alpha_3)$. It follows directly that a **volume element** is given by

$$\delta V = h_1\,h_2\,h_3\,\delta\alpha_1\,\delta\alpha_2\,\delta\alpha_3\,.$$

The **volume integral** for $f = f(\alpha_1,\alpha_2,\alpha_3)$ is then

$$\int_V f\,\mathrm{d}V = \iiint f \quad h_1\,h_2\,h_3\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3. \tag{4.10}$$

Note that $\delta$ denotes a small but finite element, and that the infinitesimal is denoted $\mathrm{d}$ in the integral.

Consider a surface $S$ defined by $\alpha_1 = const$. A **surface element** is generated by variations in the other two variables in this case. Hence,

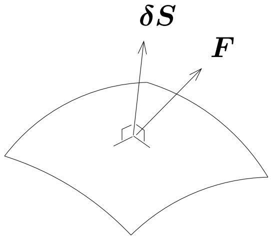
$$\delta S = h_2\,h_3\,\delta\alpha_2\,\delta\alpha_3\,.$$

The **surface integral** for scalar $g = g(\alpha_1,\alpha_2,\alpha_3)$ in this case is

$$\int_S g\,\mathrm{d}S = \iint g \quad h_2\,h_3\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3. \tag{4.11}$$

(In previous courses the prefactors $h_2\,h_3$ etc. may have been derived via the Jacobian determinant. This is easier.)

Often we want to know the **flux** of a vector field passing through a surface.




The **vector surface element** $\boldsymbol{\delta S} = \hat{\boldsymbol{n}}\,\delta S$, where $\hat{\boldsymbol{n}}$ is a **normal to the surface** $S$. We may now consider the component of a vector field $\boldsymbol{F}$ in the direction of $\hat{\boldsymbol{n}}$ and write

$$\int_S \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{S} \;=\; \int_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \;=\; \int_S |\boldsymbol{F}|\cos\theta\,\mathrm{d}S\,. \tag{4.12}$$

The rightmost form should be used with care – there $\theta$ is the angle between $\boldsymbol{F}$ and the unit vector $\hat{\boldsymbol{n}}$, NOT to be confused with $\theta$ of a coordinate system.

Convention is to take $\boldsymbol{\delta S}$ to be an **outwards normal for a closed surface** $S$, i.e. when $S$ encloses a volume $V$. If $S$ is not closed then the direction should be specified.

**Example 4.4.1 (Volume integral)**
$V$ is a cylinder bounded by $r = a$, $z = 0$ and $z = h$. Find $I = \int_V 2z(z^2 + r^2)\,\mathrm{d}V$.

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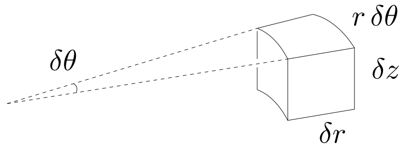
$V \equiv$ cylinder $\rightarrow$ CPs: $h_1 = h_3 = 1$, $h_2 = r$.
$\Rightarrow \quad \delta V = r\,\delta r\,\delta\theta\,\delta z$.




$$\begin{aligned} I &= \int_V 2z(z^2+r^2)\,\mathrm{d}V \;=\; 2\int_{z=0}^{h}\int_{\theta=0}^{2\pi}\int_{r=0}^{a}(z^3+zr^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\ &= 4\pi\int_{z=0}^{h}\left[z^3\frac{r^2}{2}+z\frac{r^4}{4}\right]_{r=0}^{a}\mathrm{d}z \\ &= 4\pi\left(\frac{h^4}{4}.\frac{a^2}{2}+\frac{h^2}{2}.\frac{a^4}{4}\right) \;=\; \frac{\pi}{2}h^2a^2(h^2+a^2)\,. \end{aligned}$$

**Example 4.4.2 (Surface integral)**
Evaluate $\int_S \boldsymbol{r}\cdot\mathbf{d}\boldsymbol{S}$ where $S$ is the spherical surface define by $r = a$.

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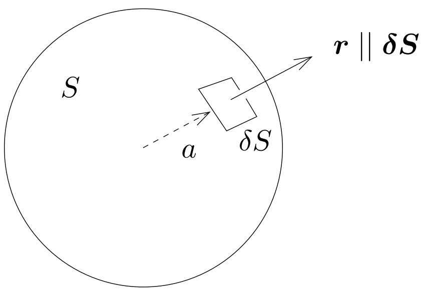



$r = a \;\rightarrow$ SPs: $\alpha_1 = const,\; h_2 = r,\; h_3 = r\sin\theta$.
$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}, \quad \boldsymbol{\delta S} = r^2\sin\theta\,\delta\theta\,\delta\phi\,\hat{\boldsymbol{r}}$,
so $\boldsymbol{r}$ is parallel to $\boldsymbol{\delta S}$ in this case.
Therefore, $\boldsymbol{r}\cdot\boldsymbol{\delta S} = a^3\sin\theta\,\delta\theta\,\delta\phi$ on $S$.

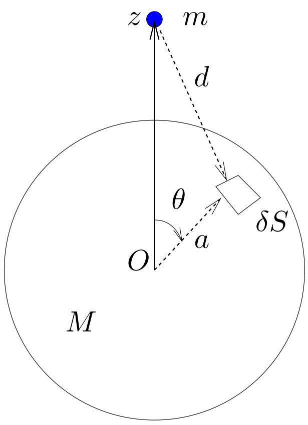
$$\begin{aligned} \int_S \boldsymbol{r}\cdot\mathbf{d}\boldsymbol{S} &= \int_0^{2\pi}\int_0^{\pi} a^3\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= 2\pi\,a^3\int_0^{\pi}\sin\theta\,\mathrm{d}\theta \;=\; 4\,\pi\,a^3\,. \end{aligned}$$

**Worked Example 4.4**

(A) Find the surface area of a sphere of radius $a$.

(B) A spherical shell has inner radius $a$, outer radius $b$ and density profile $\rho(r) = \rho_0\, a/r$, where $\rho_0$ is a constant. Find its mass $m$.

**Example 4.4.3 (The Shell Theorem)**




Consider a thin spherical shell of radius $a$ and uniform mass per unit area $\sigma$.

$\Rightarrow$ total mass of shell $M = 4\pi a^2\,\sigma$.

$\Rightarrow$ mass of surface element $\delta M = \sigma\,\delta S$.

Let mass $m$ be a distance $z$ from the centre of the shell. Then with $\boldsymbol{z} = z\,\boldsymbol{k}$, $\boldsymbol{a} = a\,\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{r}}\cdot\boldsymbol{k} = \cos\theta$, we have

$$\boldsymbol{z} + \boldsymbol{d} = \boldsymbol{a} \quad\Rightarrow\quad \boldsymbol{d} = \boldsymbol{a} - \boldsymbol{z}$$

$$\Rightarrow\quad d^2 = (\boldsymbol{a}-\boldsymbol{z})\cdot(\boldsymbol{a}-\boldsymbol{z}) = \boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{a}\cdot\boldsymbol{a}-2\boldsymbol{a}\cdot\boldsymbol{z} = a^2+z^2-2az\cos\theta\,.$$

The contribution to the gravitational potential for each small surface element is

$$\delta V = -\frac{G\,m\,\delta M}{d}\,,\quad \text{and}\quad \delta M = \sigma\,\delta S\,,\quad \delta S = r^2\sin\theta\,\delta\theta\,\delta\phi\,,\quad \text{with}\quad r = a\,.$$

The total potential is then $V = \sum\delta V = -G\,m\,\sigma\sum(\delta S/d)$

$$\Rightarrow\quad V = -Gm\sigma\int_0^{2\pi}\!\!\int_0^{\pi}\frac{a^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}{d} \;=\; -2\pi a^2\sigma\,Gm\int_0^{\pi}\frac{\sin\theta\,\mathrm{d}\theta}{\sqrt{a^2+z^2-2az\cos\theta}}\,.$$

Let $u = \cos\theta$, $\quad \mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$

$$\Rightarrow\quad V = -\tfrac{1}{2}MGm\int_{-1}^{1}\frac{\mathrm{d}u}{\sqrt{a^2+z^2-2azu}} \;=\; \tfrac{1}{2}GMm\left[\frac{1}{az}\sqrt{a^2+z^2-2azu}\right]_{-1}^{1}$$

$$\sqrt{a^2+z^2\mp 2az} = \sqrt{(a\mp z)^2} = |a\mp z|\,,\quad a, z > 0$$

$$\Rightarrow\quad V = \frac{GMm}{2az}\,\{|a-z| - (a+z)\}$$

If $z \le a$, then $|a-z| = (a-z) \;\Rightarrow\; \{(a-z)-(a+z)\} = -2z$.

If $z \ge a$, then $|a-z| = (z-a) \;\Rightarrow\; \{(z-a)-(a+z)\} = -2a$.

Finally we have the result! :

If $z \le a,$ then $V = -\dfrac{GMm}{a} = const \;\Rightarrow$ inside the sphere, force $\boldsymbol{F} = -\boldsymbol{\nabla}V = \boldsymbol{0}\,.$

If $z \ge a,$ then $V = -\dfrac{GMm}{z} \;\Rightarrow$ outsize the sphere, $\boldsymbol{F}$ same as for point mass $M$ at origin.

## 4.5 Gauss' Theorem (or the Divergence Theorem)

Let $\boldsymbol{F}$ be a vector field, and $S$ be a closed surface, enclosing a volume $V$. Then **Gauss' Theorem** states that

$$\int_S \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{F}\, \mathrm{d}V. \tag{4.13}$$

As $S$ is closed, $\mathbf{d}\boldsymbol{S}$ is the outwards normal. The theorem describes the balance between the **flux** through the surface, on the LHS, and the total divergence within a volume, on the RHS.

**Worked Example 4.5**
Verify the theroem holds where
(A) $V$ is a sphere of radius $a$, centred on the origin, and $\boldsymbol{F} = \boldsymbol{r}$.
(B) $V$ is the cylinder bounded by $r = a$, $z = 0$, $z = h$, and $\boldsymbol{F} = xz^3\boldsymbol{i} + yz^3\boldsymbol{j} + z^2(x^2 + y^2)\boldsymbol{k}$.

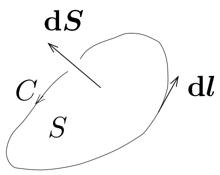
### Corollary of Gauss' Theorem

Let $\boldsymbol{F} = \phi\boldsymbol{a}$, where $\phi$ is a scalar field and $\boldsymbol{a}$ is an arbitrary constant vector. Expanding $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \boldsymbol{a} \cdot \boldsymbol{\nabla}\phi + \phi\,\boldsymbol{\nabla} \cdot \boldsymbol{a}$, but $\boldsymbol{\nabla} \cdot \boldsymbol{a} = 0$ as $\boldsymbol{a}$ is constant. Substitution into Gauss' Theorem gives $\boldsymbol{a} \cdot \int \phi\, \mathbf{d}\boldsymbol{S} = \boldsymbol{a} \cdot \int \boldsymbol{\nabla}\phi\, \mathrm{d}V$. As $\boldsymbol{a}$ is abitrary we have

$$\int_S \phi\, \mathbf{d}\boldsymbol{S} = \int_V \boldsymbol{\nabla}\phi\ \mathrm{d}V \tag{4.14}$$

Note that this is a set of three equations, one for each component of $\mathbf{d}\boldsymbol{S}$ and $\boldsymbol{\nabla}\phi$.

## 4.6 Stokes' Theorem




Let $S$ be an *open* surface, bounded by the closed circuit $C$. Then **Stokes' Theorem** states that

$$\oint_C \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{l} = \int_S \boldsymbol{\nabla} \times \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{S}. \tag{4.15}$$

**Worked Example 4.6**
Verify Stokes' Theorem for Example 3.2.2 .

**Example 4.6.1**
Show that work done $\int_A^B \boldsymbol{F} \cdot \mathbf{d}\boldsymbol{l}$ is independent of the path from $A$ to $B$ for a conservative force using Stokes' Theorem.

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As $\boldsymbol{F} = -\boldsymbol{\nabla}V$, **a conservative force is irrotational**: $\boldsymbol{\nabla} \times \boldsymbol{F} = -\boldsymbol{\nabla} \times \boldsymbol{\nabla}V = \mathbf{0}$.
Let $C_1$ and $C_2$ be two different paths from $A$ to $B$, let $C$ be the closed path $C_1 - C_2$ and $S$ be a surface enclosed by $C$. Then

$$\oint_C \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l} = \int_S \boldsymbol{\nabla}\times\boldsymbol{F}\cdot\mathbf{d}\boldsymbol{S} = 0 \quad\Rightarrow\quad \oint_C \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l} = \int_{C_1} \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l} - \int_{C_2} \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l} = 0 \quad\Rightarrow\quad \int_{C_1} \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l} = \int_{C_2} \boldsymbol{F}\cdot\mathbf{d}\boldsymbol{l}$$

Hence the integrals along different paths $C_1$ and $C_2$ from $A$ to $B$ are the same. Given that the paths are arbitrary, the work done must be independent of the path.