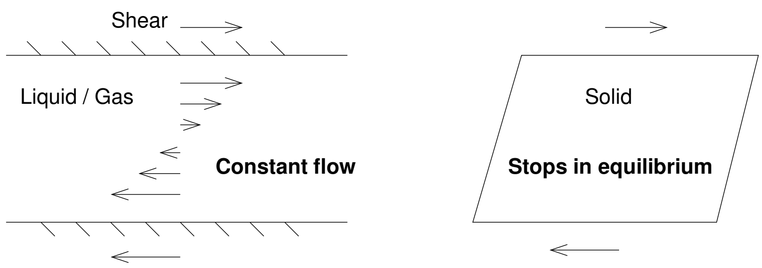
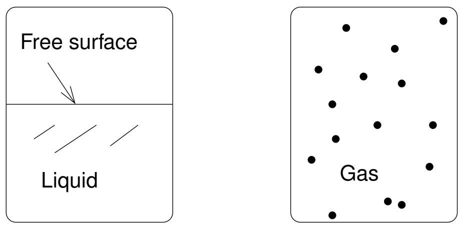
Kinematics refers to the description of motion, neglecting the origin of any forces that cause the motion.

5.1 What is a fluid?

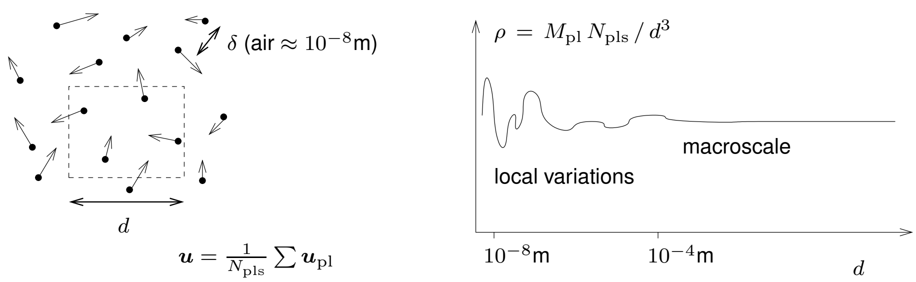
- Any medium that **flows** can be considered as a fluid. Commonly this encompasses both **liquids** and **gasses**.
- Solids behave differently from fluids when a **shear** is applied.



- There are two main distinctions between liquids and gasses:
  (a) Liquids often form a **“free” surface**, whereas gasses fill a container.
  (b) In response to changes in **pressure**, gasses are much more **compressible**, i.e. the volume of gas changes.



5.2 The continuum hypothesis

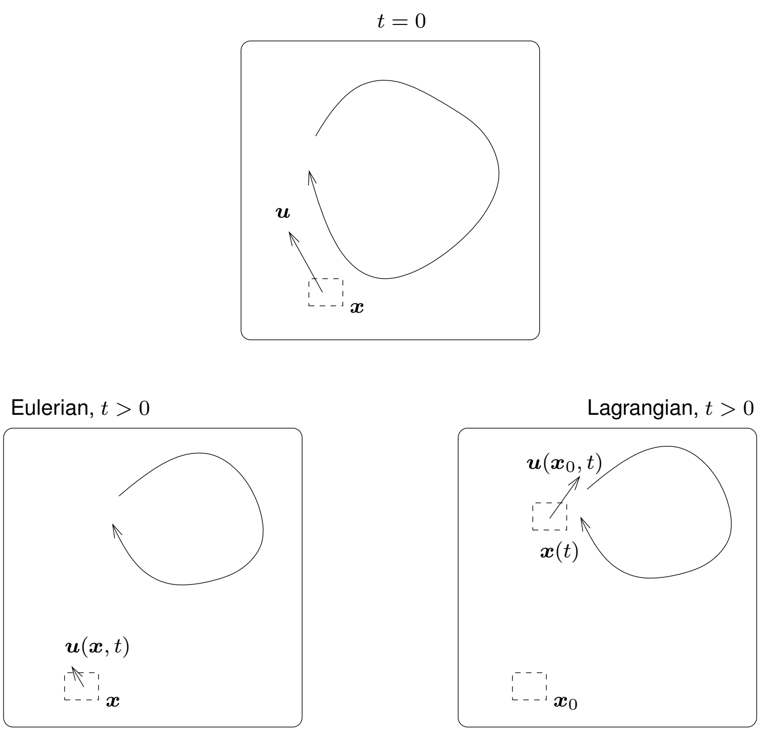


- Fluids are composed of many constantly moving and colliding molecules.
- It is impractical to model all molecules individually.

- **Continuum hypothesis**: There is a well-defined length scale, d, above which the local density, $\rho(\boldsymbol{x})$, is a well-behaved function of position, $\boldsymbol{x} = (x, y, z)$. On these **macroscopic** length-scales the medium can be treated as a continuum, and the velocity field, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, is likewise well-behaved.
  "Well-behaved" $\Rightarrow$ continuous and many-times differentiable.
  "Well-defined" $\Rightarrow$ can be measured reliably.
- Normally $d \ll D$, where D is a characteristic length scale for the flow domain. This approximation breaks down for *nano*fluidics.
- By $\boldsymbol{u}$ we refer to the motion of a **fluid element**, a group of fluid particles for which $\delta \ll d \ll D$.
- The (statistical) properties of a fluid element are determined by applied forces, geometry (boundaries, etc.), temperature and so on.

5.3 Eulerian & Lagrangian descriptions of motion

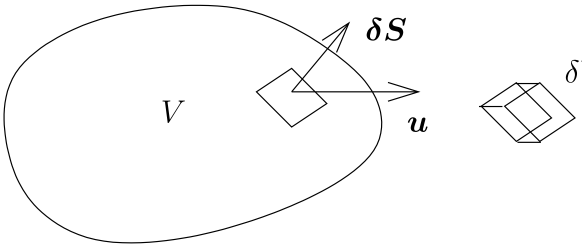
- L. Euler, 1707-1783; J-L. Lagrange, 1736-1813.
- The **Eulerian description** considers the change velocity of fluid elements at each **fixed** location, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$.
- The **Lagrangian description** considers the velocity of each fluid element **as it moves** within the fluid, $\boldsymbol{x} = \boldsymbol{x}(t)$, as a function of its initial position, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}_0, t)$, $\boldsymbol{x}_0 = \boldsymbol{x}(t = 0)$.



- Both descriptions have been attributed to Euler (1755,1759)!
- The Eulerian description is more frequently used, and will be adopted for the rest of this course.

6 *CONTINUITY, VORTICITY AND THE STREAM FUNCTION*

6.1 Conservation of mass and continuity



$$\delta V = \delta t\, \boldsymbol{u}\cdot\boldsymbol{\delta S}, \quad \delta m = \delta t\, \rho\, \boldsymbol{u}\cdot\boldsymbol{\delta S}$$

The change in mass in an arbitrary volume will be equal to the mass that has flowed through its surface,

$$\int_V \rho(\boldsymbol{x}, t+\delta t)\, \mathrm{d}V - \int_V \rho(\boldsymbol{x}, t)\, \mathrm{d}V = -\delta t \int_S \rho\, \boldsymbol{u}\cdot \mathbf{d}\boldsymbol{S}\,.$$

$\times 1/\delta t$ and using Gauss' Theorem

$$\begin{aligned}\int_V \frac{\partial \rho}{\partial t}\, \mathrm{d}V &= -\int_S \rho\, \boldsymbol{u}\cdot \mathbf{d}\boldsymbol{S} \\ &= -\int_V \boldsymbol{\nabla}\cdot(\rho\, \boldsymbol{u})\, \mathrm{d}V\end{aligned}$$

$$\text{i.e.} \quad \int_V \left\{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\, \boldsymbol{u})\right\} \mathrm{d}V = 0\,.$$

This holds for arbitrary V and therefore implies the **equation of mass conservation**:

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\, \boldsymbol{u}) = 0\,.} \tag{6.1}$$

For an **incompressible** fluid $\rho = const$. Usually, for sub-sonic flows $\rho = const$ is a very accurate approximation. For this case the **continuity equation** follows:

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,.} \tag{6.2}$$

6.2 Vorticity and circulation

The **vorticity** is the curl of the velocity field, usually denoted $\boldsymbol{\omega}$,

$$\boxed{\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}.} \tag{6.3}$$

Note that $\boldsymbol{\omega}$ is always divergence-free: $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = \mathbf{0}$.
Let C be a closed circuit lying entirely within the fluid. The **circulation**, Γ, around C is

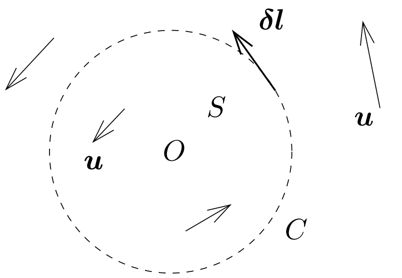
$$\boxed{\Gamma = \oint_C \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{l}.} \tag{6.4}$$

Provided $\boldsymbol{u}$ has no singularities on the surface S enclosed by C, then vorticity and circulation are related via Stokes' Theorem (4.6):

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{l} = \int_S \boldsymbol{\nabla} \times \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{S} = \int_S \boldsymbol{\omega} \cdot \mathbf{d}\boldsymbol{S}.$$

Example 6.2.1 (Solid-body rotation)
$\boldsymbol{u} = \Omega \times \boldsymbol{r}$ represents rotation about the origin, with axis in the direction of $\boldsymbol{\Omega}$. It's vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times (\Omega \times \boldsymbol{r}) = 2\,\boldsymbol{\Omega}$ (see Worked Example 5.5(G)) and it is divergence-free:

$$\boldsymbol{\nabla} \cdot (\Omega \times \boldsymbol{r}) = \partial_i\, \epsilon_{ijk}\, \Omega_j\, x_k = \epsilon_{ijk}\, \Omega_j \underbrace{\partial_i\, x_k}_{\delta_{ik},\, k \to i} = \epsilon_{iji}\, \Omega_j = 0\,.$$



Circulation on a circular circuit C (CPs):

On C, $\boldsymbol{\delta l} = r\,\delta\theta\,\hat{\boldsymbol{\theta}}$, $\boldsymbol{u} = \Omega r \hat{\boldsymbol{\theta}}$, and $r = a$,

$$\Rightarrow \quad \Gamma = \oint_C \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{l} = \int_0^{2\pi} \Omega a^2 \mathrm{d}\theta = 2\pi\Omega a^2\,.$$

On S, $\boldsymbol{\delta S} = \delta S\, \boldsymbol{k}$, $\boldsymbol{\omega} = 2\Omega\, \boldsymbol{k}$,

$$\Rightarrow \quad \Gamma = \int_S \boldsymbol{\omega} \cdot \mathbf{d}\boldsymbol{S} = 2\Omega \int_S \mathrm{d}S = 2\Omega\pi a^2\,.$$

Worked Example 6.2 (The line vortex)
Calculate the circulation for

$$\boldsymbol{u} = \frac{\kappa}{2\pi r}\,\hat{\boldsymbol{\theta}} \quad \text{(in CPs).} \tag{6.5}$$

Note that the surface integral cannot be integrated trivially.

6.3 The stream function and streamlines for 2D flow

If we write (see (3.6))

$$\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{A}, \tag{6.6}$$

then the velocity field satisfies

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \ \text{ automatically.}$$

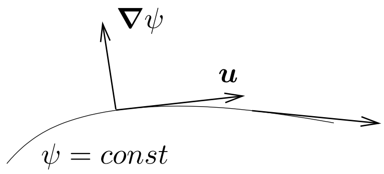
For a **two-dimensional (2D) flow**, let $\boldsymbol{k}$ be perpendicular to the plane of motion. Taking $\boldsymbol{A} = \psi\boldsymbol{k}$, then we have that (in Cartesians)

$$\boldsymbol{u} \ = \ \boldsymbol{\nabla} \times (\psi\boldsymbol{k}) \ = \ (u,\, v,\, 0) \quad \Rightarrow \quad u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}\,. \tag{6.7}$$

The scalar $\psi = \psi(x, y)$ is called the **stream function**.

Now, $\boldsymbol{\nabla}\psi$ is perpendicular to isocontours given by $\psi = const$, and $\boldsymbol{\nabla}\psi$ is perpendicular to $\boldsymbol{u}$, since

$$\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} = 0\,.$$



The isocontours given by $\psi = const$ must therefore be parallel to $\boldsymbol{u}$; we call them **streamlines**. Further, from (6.7) we have that

$$|\boldsymbol{u}| = |\boldsymbol{\nabla}\psi|. \tag{6.8}$$

Example 6.3.1 (Stagnation point flow)

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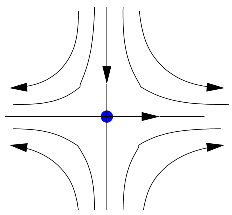

Streamfunction $\psi = kxy \quad \Rightarrow \quad \boldsymbol{u} = k(x, -y, 0)\,.$

Contours $\psi = const \quad \Rightarrow \quad y = \frac{A}{x}, \quad A\ const.$

There is a a *stagnation point* at $x = y = 0$ where $\boldsymbol{u} = \boldsymbol{0}.$

Worked Example 6.3 (Flow around a cylinder)
Convert the following stream function to CPs and sketch streamlines:

$$\psi = Uy - \frac{Ua^2 y}{x^2 + y^2}\,.$$