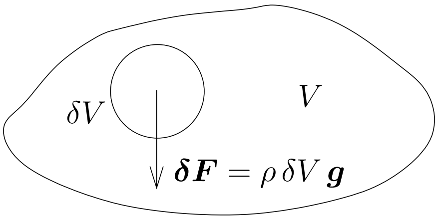
# 7 *EQUATIONS OF MOTION FOR AN INVISCID FLUID*

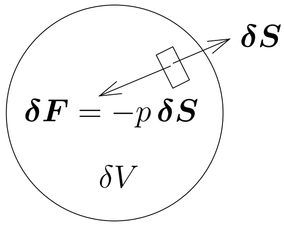
**Viscosity** is a measure of the "**thickness**" of a fluid, and its **resistance to shearing** motions. Honey is difficult to stir because of its high viscosity, whereas water flows easily, having much lower viscosity. Fluids are often modelled as though they were completely **inviscid**.

## 7.1 Force and acceleration on a volume

### *7.1.1 Body and surfaces forces*

There are two type of forces that may act on a fluid element, body forces and surface forces. A **body force** acts throughout the volume, on all elements, $\delta V$. A good example is the gravitational force, $\delta m\, \boldsymbol{g} = \rho\, \delta V\, \boldsymbol{g}$ (left-hand sketch).




Now consider an imaginary surface $S$ separating two regions within the fluid (right-hand sketch). The motions in one region of the fluid can exert a force on the other region. The **surface force** is proportional to the size of the area, $\delta S$, and for an inviscid fluid it acts only in the direction $\boldsymbol{\delta S}$. (The regions of fluid push each other around but slip freely against each-other.) This is the **inviscid approximation**. Inserting a local constant of proportionality $p$, we write that the force on the surface is

$$-p\, \boldsymbol{\delta S},$$

where $p = p(\boldsymbol{x}, t)$ is the **pressure**.

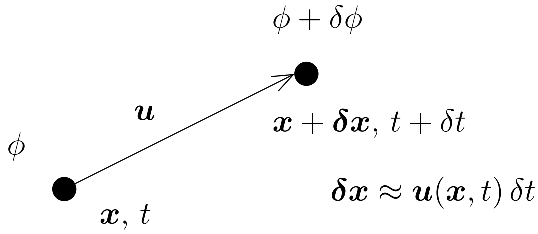
Note that if the volume $\delta V$ is replaced with a solid, then the inviscid approximation implies a **free-slip boundary condition** at the interface with the solid, and the **total pressure force on the solid** is

$$\boxed{\boldsymbol{F} = -\int_S p\, \mathbf{d}\boldsymbol{S}\,,} \qquad (7.1)$$

where $\boldsymbol{\delta S}$ points into the fluid.

### *7.1.2 Rate of change, when moving with a fluid*

Consider the change in an arbitrary scalar $\phi$ associated with a fluid element as it moves with the flow:




Define $\dfrac{D}{Dt}\phi \equiv \lim_{\delta t \to 0} \dfrac{\delta\phi}{\delta t}$.

The Taylor expansion of a scalar $\phi = \phi(x, y, z, t)$ gives

$$\delta\phi = \frac{\partial\phi}{\partial x}\,\delta x + \frac{\partial\phi}{\partial y}\,\delta y + \frac{\partial\phi}{\partial z}\,\delta z + \frac{\partial\phi}{\partial t}\,\delta t \;=\; \delta t\,\frac{\partial\phi}{\partial t} + (\delta t\boldsymbol{u})\cdot\boldsymbol{\nabla}\phi$$

$$\Rightarrow \qquad \boxed{\frac{D}{Dt}\phi = \left\{\frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\right\}\,\phi\,.} \tag{7.2}$$

We call $D/Dt = \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ the **material derivative**, which gives the rate of change of a variable seen from a point *moving with the flow*. The **acceleration** on a fluid element is then

$$\frac{D\boldsymbol{u}}{Dt} = \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\,.$$

A flow is said to be **steady** when $\partial_t\boldsymbol{u} = \boldsymbol{0}$, i.e. when it does not change in time. The term $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$, however, need not be zero for a steady flow.

## 7.2 Euler's equation for inviscid fluid motion

In the previous section we calculated the accelerations and forces acting on a fluid element. We are finally in a position to relate them through Newton's second law:

$$m\boldsymbol{a} = \boldsymbol{F} \quad \Rightarrow \quad \int_V \rho\,\frac{D\boldsymbol{u}}{Dt}\,\mathrm{d}V = \int_V \rho\,\boldsymbol{g}\,\mathrm{d}V - \int_S p\,\mathbf{d}\boldsymbol{S}\,.$$

Applying the corollary to Gauss' Theorem (4.14) to the last term gives

$$\int_V \rho\frac{D\boldsymbol{u}}{Dt}\,\mathrm{d}V = \int_V (\rho\,\boldsymbol{g} - \boldsymbol{\nabla}p)\,\mathrm{d}V\,.$$

As the volume element considered is arbitrary, we have **Euler's equation**

$$\boxed{\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} \;=\; \boldsymbol{g} - \frac{1}{\rho}\,\boldsymbol{\nabla}p\,.} \tag{7.3}$$

This is a set of three equations, one for each component, but the three components of $\boldsymbol{u}$ plus $p$ constitute four unknowns. We require either an *equation of state*, which links pressure to density (and possibly temperature), *or*, if the fluid is incompressible, the *continuity equation* $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ (6.2) provides our fourth governing equation.

The **nonlinear term**, $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$, makes solutions difficult to find, but is also responsible for a wide variety of possible dynamics.

### *7.2.1 Boundary conditions*

The natural boundary condition to accompany Euler's equation is usually the **no-penetration condition**, i.e. no flow through the boundary. For a stationary boundary with normal $\hat{\boldsymbol{n}}$, no-penetration is expressed as $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = 0$. If the boundary moves with velocity $\tilde{\boldsymbol{U}}$, the condition becomes $\boldsymbol{u}\cdot\hat{\boldsymbol{n}} = \tilde{\boldsymbol{U}}\cdot\hat{\boldsymbol{n}}$.

Another possible 'boundary condition' is that there is a **background flow**, far from the region of interest, i.e. $\boldsymbol{u}\to\boldsymbol{U}$ "at infinity" (e.g. as $r\to\infty$).

**Worked Example 7.2 (Flow around a sphere)**
Check that the velocity field

$$\boldsymbol{u} = U\left\{(1-\frac{a^3}{r^3})\cos\theta\,\hat{\boldsymbol{r}} - (1+\frac{a^3}{2r^3})\sin\theta\,\hat{\boldsymbol{\theta}}\right\}$$

(in SPs) represents an incompressible flow and check the boundary conditions.

### *7.2.2 Kinetic Energy*

Using the vector calculus identity $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}) - \boldsymbol{u}\times\boldsymbol{\omega}$ (E.9), where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$, and assuming $\rho$ is constant, Euler's equation (7.3) may be rewritten

$$\frac{\partial\boldsymbol{u}}{\partial t} = \boldsymbol{u}\times\boldsymbol{\omega} - \boldsymbol{\nabla}\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + \frac{p}{\rho} - \boldsymbol{g}\cdot\boldsymbol{r}\right). \tag{7.4}$$

This is sometimes called the **rotational form**. (Note that conservative forces can always be written in the form $\boldsymbol{F} = -\boldsymbol{\nabla}V$. Here, constant $\boldsymbol{g} = \boldsymbol{\nabla}(\boldsymbol{g}\cdot\boldsymbol{r})$, where $\boldsymbol{r}$ is the position vector.)

To determine the time dependence of the Kinetic Energy (KE), we multiply (7.4) through by $\rho$, take the dot product with $\boldsymbol{u}$ and integrate,

$$\rho\int\boldsymbol{u}\cdot\partial_t\boldsymbol{u}\,\mathrm{d}V \;=\; \rho\int\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{\omega})\,\mathrm{d}V - \int\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}V, \tag{7.5}$$

where $\psi$ is the term in brackets in (7.4). On the left hand side $\boldsymbol{u}\cdot\partial_t\boldsymbol{u} = \frac{1}{2}\partial_t(\boldsymbol{u}\cdot\boldsymbol{u})$, i.e. we have

$$\partial_t\left(\tfrac{1}{2}\rho\int\boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}V\right) \;=\; \partial_t(\text{KE}) \tag{7.6}$$

(cf. $\frac{1}{2}mv^2$). On the right hand side of (7.5), $\boldsymbol{u}\times\boldsymbol{\omega}$ is perpendicular to $\boldsymbol{u}$, which implies that $\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{\omega}) = 0$. For the last term we use identity $\boldsymbol{\nabla}\cdot(\psi\boldsymbol{u}) = \boldsymbol{u}\cdot\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\cdot\boldsymbol{u}$ (E.6), with $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ for an incompressible flow. Then, using Green's Theorem,

$$\int\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi\,\mathrm{d}V \;=\; \int\boldsymbol{\nabla}\cdot(\psi\boldsymbol{u})\,\mathrm{d}V \;=\; \int\psi\boldsymbol{u}\cdot\mathbf{d}\boldsymbol{S}.$$

Finally, we have that

$$\partial_t(\text{KE}) \;=\; -\int\psi\,\boldsymbol{u}\cdot\mathbf{d}\boldsymbol{S}. \tag{7.7}$$

This implies that the kinetic energy in the flow is influenced only by what happens at the boundary surface (provided that background forces are conservative). We have shown that there is **no internal dissipation of energy for an inviscid fluid**. If there is no penetration and the walls are stationary, then $\boldsymbol{u}\cdot\mathbf{d}\boldsymbol{S} = 0$ and the kinetic energy remains constant.

# 8 *IRROTATIONAL FLOW*

For an **irrotational** field

$$\boxed{\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}.} \tag{8.1}$$

But do such flows arise? Observing that $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \boldsymbol{0}$ for any scalar $\phi$, from the curl of (7.4)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}).$$

This implies Lagrange's Theorem: "A body of inviscid fluid in **irrotational flow continues to move irrotationally**", i.e. $\boldsymbol{\omega} = \boldsymbol{0} \;\Rightarrow\; \partial_t \boldsymbol{\omega} = \boldsymbol{0}$. In practice, viscosity leads to dissipation of vorticity in the interior of a flow, and vorticity is generated only at the boundaries.

## 8.1 The velocity potential for irrotational flow

If we write

$$\boxed{\boldsymbol{u} = \boldsymbol{\nabla}\phi,} \tag{8.2}$$

then the flow satisfies

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0} \;\text{ automatically (see (3.5)).}$$

Irrotational flow is therefore sometimes called **potential flow**. By continuity (6.2), the **velocity potential** satisfies the Laplace equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \nabla^2\phi = 0\,. \tag{8.3}$$

Solutions for inviscid flow are therefore solutions of the Laplace equation, subject to boundary conditions

$$\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \tilde{\boldsymbol{U}} \cdot \hat{\boldsymbol{n}}\,, \tag{8.4}$$

where $\tilde{\boldsymbol{U}}$ is the velocity of the boundary. Solutions to linear equations like (8.3) are often determined using the *method of separation of variables*.

**Example 8.1.1 (Flow around a sphere, continued.)**
The flow

$$\boldsymbol{u} = \boldsymbol{\nabla}W, \qquad W = U\left(r\cos\theta + \frac{a^3\cos\theta}{2r^2}\right) \quad \text{in SPs}$$

corresponds to that of Worked Example 7.2. It was shown that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, and therefore $W$ satisfies the Laplace equation: $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}W = \nabla^2 W = 0\,.$

**Worked Example 8.1**
Find $\phi$ for the line vortex of Worked Example 6.2.

## 8.2 Bernoulli's Integral for irrotational flow

Euler's equation rotational form (7.4) with $\boldsymbol{\omega} = \mathbf{0}$ gives

$$\boldsymbol{\nabla}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + \frac{p}{\rho} - \boldsymbol{g}\cdot\boldsymbol{r}\right) = 0.$$

Upon integration we therefore have that

$$\frac{p}{\rho} = \boldsymbol{g}\cdot\boldsymbol{r} - \frac{\partial \phi}{\partial t} - \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + const\,, \tag{8.5}$$

where the combination $p/\rho$ is known as **Bernoulli's Integral**. As we have integrated $\boldsymbol{\nabla}$, the constant term could be an arbitrary function of time. It can be absorbed into $\phi$, however, without changing $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. (Take $\phi' = \phi - \int_0^t f(s)\,\mathrm{d}s$.)

For flow in a small area ($|\boldsymbol{r}|$ small), or for very fast flow, or flow perpendicular to $\boldsymbol{g}$, we have $|\boldsymbol{g}\cdot\boldsymbol{r}| \ll |\boldsymbol{u}\cdot\boldsymbol{u}|$. If it is also steady, $\partial_t\boldsymbol{u} = \mathbf{0}$, then

$$\frac{p}{\rho} \approx -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + const\,, \tag{8.6}$$

i.e. high speed $\longleftrightarrow$ low pressure.

**Example 8.2.1 (Lift for a simple wing)**

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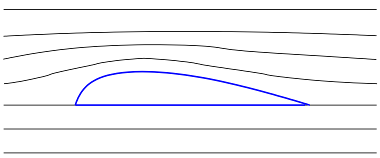


$\psi = const$ on each streamline (section 6.3).

Streamlines squashed near upper surface of wing

$\Rightarrow$ increased $|\boldsymbol{\nabla}\psi|$

$\Rightarrow$ higher $|\boldsymbol{u}|$ by (6.8)

$\Rightarrow$ lower pressure on upper surface by (8.6).

**Example 8.2.2 (Many more examples!)**

Try: Youtube "Bernoulli's principle - physics experiment"
https://www.youtube.com/watch?v=P-xNXrELCmU

## 8.3 Flow around cylinders

A solution for flow around a cylinder of radius $a$ is given by a stream function (Worked Example 6.3),

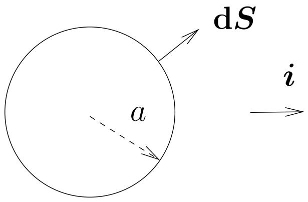
$$\boldsymbol{u} = \boldsymbol{\nabla}\times(\psi\,\hat{\boldsymbol{z}})\,, \qquad \psi = Uy - \frac{Ua^2y}{x^2+y^2} = U\left(r - \frac{a^2}{r}\right)\sin\theta\,,$$

$$\Rightarrow \quad \boldsymbol{u} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\hat{\boldsymbol{r}} - U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\hat{\boldsymbol{\theta}}. \tag{8.7}$$

It also has velocity potential,

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi\,, \qquad \phi = U\left(r + \frac{a^2}{r}\right)\cos\theta\,, \tag{8.8}$$

and the flow is therefore irrotational. The component of force on the cylinder in the direction of the background flow, $U\boldsymbol{i}$, or **drag**, from (7.1) is




$$\boldsymbol{F}\cdot\boldsymbol{i} = -\int p\,\boldsymbol{i}\cdot\mathbf{d}\boldsymbol{S} = -\int\int p\cos\theta\, r\,\mathrm{d}\theta\,\mathrm{d}z \qquad (8.9)$$

$$\Rightarrow \text{ Drag per unit length } F_d = -\int_0^{2\pi} pa\cos\theta\,\mathrm{d}\theta\,.$$

Ignoring gravity, Bernoulli's Integral implies

$$\frac{p}{\rho} = -\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + const$$

If $p\to p_\infty$ as $r\to\infty$, then $const = \frac{p_\infty}{\rho}+\frac{1}{2}U^2$, and

$$p = p_\infty - \frac{1}{2}\rho\,\boldsymbol{u}\cdot\boldsymbol{u} + \frac{1}{2}\rho U^2,$$

i.e. we have an expression for the pressure everywhere, with, from (8.7),

$$\boldsymbol{u}\cdot\boldsymbol{u} = U^2\cos^2\theta\left(1-\frac{a^2}{r^2}\right)^2 + U^2\sin^2\theta\left(1+\frac{a^2}{r^2}\right)^2 = U^2\left(1+\frac{a^4}{r^4}\right) + U^2\frac{2a^2}{r^2}\underbrace{(\sin^2-\cos^2)}_{-\cos 2\theta}$$

For the drag we need consider only $r=a$,

$$p = p_\infty + \frac{1}{2}\rho U^2(2\cos 2\theta - 1) = p_\infty + \frac{1}{2}\rho U^2(1-4\sin^2\theta)$$

$$\Rightarrow F_d = -(p_\infty + \frac{1}{2}\rho U^2)\,a\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta + 2\rho U^2 a\int_0^{2\pi}\sin^2\theta\cos\theta\,\mathrm{d}\theta\,.$$

But $\int_0^{2\pi}\cos\theta\,\mathrm{d}\theta$ and $\int_0^{2\pi}\sin^2\theta\cos\theta\,\mathrm{d}\theta = [\frac{1}{3}\sin^3\theta]_0^{2\pi} = 0 \Rightarrow$ **zero drag!**

Everyday experience tells us that in the presence of a background flow there should be a drag on the cylinder, and hence the result is wrong. This was known as **d'Alembert's paradox** (1752).

In reality, the flow given by $\phi$ is reasonable, but close to the cylinder viscosity is important and leads to drag.

**Worked Example 8.3**
We may add solutions since

$$\nabla^2\phi_1 = 0,\ \ \nabla^2\phi_2 = 0 \quad\Rightarrow\quad \nabla^2(\phi_1+\phi_2) = 0\,.$$

Adding the potential for the line vortex (Worked Example 8.1) to the potential above (8.8)

$$\phi = U\left(1+\frac{a^2}{r^2}\right)\cos\theta + \frac{\kappa}{2\pi}\theta \quad\Rightarrow\quad \boldsymbol{u} = U\cos\theta\left(1-\frac{a^2}{r^2}\right)\hat{\boldsymbol{r}} + \left[-U\sin\theta\left(1+\frac{a^2}{r^2}\right)+\frac{\kappa}{2\pi r}\right]\hat{\boldsymbol{\theta}}\,.$$

The Joukowski transformation maps the circle to the cross-section of an aerofoil, and was used in **early aerofoil design**.

Locate the stagnation points and show the there is a net **lift** on the cylinder.

# 9 *DYNAMIC STABILITY OF A FLOW*

The stability of a flow refers to whether or not it is likely or not to undergo a transition to new flow pattern, e.g. from a 'laminar' (smooth) flow to chaotic turbulence.
Suppose that $\boldsymbol{U}$ and $P$ are a steady solution of the Euler's equation (7.3) for an incompressible fluid, i.e.

$$\cancel{\partial_t \boldsymbol{U}} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U} \;=\; \boldsymbol{g} - \frac{1}{\rho}\boldsymbol{\nabla}P. \tag{9.1}$$

Suppose $\boldsymbol{U}, P$ is given a small perturbation $\boldsymbol{u}' \ll \boldsymbol{U},\ \ p' \ll P$. The total flow, $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$, $p = P + p'$, follows Euler's equation (7.3), and subtracting (9.1) from (7.3) gives

$$\partial_t \boldsymbol{u}' + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{U} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \cancel{\boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}'} = -\frac{1}{\rho}\boldsymbol{\nabla}p' \qquad \text{i.e.} \qquad \partial_t \boldsymbol{u}' = L\,\boldsymbol{u}', \tag{9.2}$$

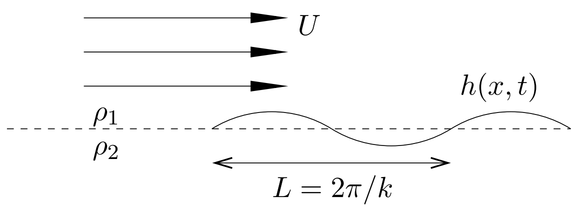
where $L$ is a constant *linear* operator. (From the divergence of (9.2), it can be shown that $p'$ is also a linear function of $\boldsymbol{u}'$.) Assuming the form

$$\boldsymbol{u}'(\boldsymbol{x},t) = \hat{\boldsymbol{u}}(\boldsymbol{x})\,\mathrm{e}^{\sigma t} \qquad \Rightarrow \qquad \sigma\,\hat{\boldsymbol{u}} = L\,\hat{\boldsymbol{u}}\,, \tag{9.3}$$

i.e. gives an **eigenvalue problem**, where the eigenvalue **$\sigma$ is the growth rate** of a disturbance. If there is an eigenvalue $\sigma$ with $\Re(\sigma) > 0$, the perturbation $\boldsymbol{u}$ will grow and the flow $\boldsymbol{U}$ is **unstable**. If all eigenvalues have $\Re(\sigma) < 0$, then $\boldsymbol{U}$ is **stable**.
The eigenfunctions $\hat{\boldsymbol{u}}(\boldsymbol{x})$ are often called the 'normal modes'. For the case of Fourier modes, each mode is assigned a wavenumber $k$, related to the wavelength $L$ by $L = 2\pi/k$.

**Worked Example 9.0 (The Kelvin–Helmholtz instability)**

Using Fourier modes, $\hat{\boldsymbol{u}}(x) \sim \mathrm{e}^{\mathrm{i}kx}$, for the setup to the right, show that the growth rate of a small disturbance, and $h(x,t)$, is given by




$$\sigma = \frac{-\rho_1 U k}{\rho_1+\rho_2}\,\mathrm{i} \pm \left\{ \left(\frac{Uk}{\rho_1+\rho_2}\right)^2 \rho_1\rho_2 + \frac{\rho_1-\rho_2}{\rho_1+\rho_2}gk \right\}^{\frac{1}{2}} .$$

Physical examples:

- **Still air over water**, put $\rho_1 = 0$, $U = 0$: Then $\sigma = \pm\mathrm{i}\sqrt{gk}$. Waves disperse, but neither grow nor decay, as $\Re(\sigma) = 0$; e.g. waves on the sea, small ripples on a pond. From $\mathrm{e}^{\sigma t + \mathrm{i}kx} = \mathrm{e}^{\mathrm{i}(kx \pm \sqrt{gk}\,t)}$ with $L = 2\pi/k$, they travel at speed $\sqrt{Lg/2\pi}$.
- **Shear instability**, same fluid, $\rho_1 = \rho_2$ but $U \neq 0$: Then $\sigma = -\frac{1}{2}Uk\mathrm{i} \pm \frac{1}{2}Uk$. Given that $\Re(\sigma) > 0$ for any $k \neq 0$, the flow is unstable.
- **More generally,** the condition for instability, $\Re(\sigma) > 0$, gives

$$\rho_1\rho_2 k U^2 > (\rho_2^2 - \rho_1^2)g,$$

i.e., **for waves to grow, $U$ must be greater than a 'critical value'**.