MAS230 | Example Sheet 4

1. $\underline{F} = (2x+2y+4z,\ 2x+3y-2z,\ 4x-2y-5z)$

By (5.1) $\nabla\cdot\underline{F} = 2+3-5 = \boxed{0}$

By (5.2) $\nabla\wedge\underline{F} = (-2-(-2))\underline{i} + (4-4)\underline{j} + (2-2)\underline{k} = \boxed{\underline{0}}$

By (5.10)

$$(\underline{F}\cdot\nabla)\underline{F} = \{(2x+2y+4z)\partial_x + (2x+3y-2z)\partial_y + (4x-2y-5z)\partial_z\}\{(2x+2y+4z)\underline{i} + (2x+3y-2z)\underline{j} + (4x-2y-5z)\underline{k}\}.$$

$$= \{2(2x+2y+4z) + 2(2x+3y-2z) + 4(4x-2y-5z)\}\underline{i}$$
$$+ \{2(2x+2y+4z) + 3(2x+3y-2z) - 2(4x-2y-5z)\}\underline{j}$$
$$+ \{4(2x+2y+4z) - 2(2x+3y-2z) - 5(4x-2y-5z)\}\underline{k}$$
$$= \boxed{(24x+2y-16z)\underline{i} + (2x+17y+12z)\underline{j} + (-16x+12y+45z)\underline{k}}$$

By (5.7) $\nabla^2\underline{F} = \boxed{\underline{0}}$

$\nabla\wedge\underline{F} = \underline{0} \Rightarrow \exists\ \varphi$ s.t. $\underline{F} = \nabla\varphi$ by (5.5)

integrating

$$\partial_x\varphi = 2x+2y+4z \implies \varphi = x^2 + 4xz + 2xy + f(y,z)$$
$$\partial_y\varphi = 2x+3y-2z \implies \varphi = \tfrac{3}{2}y^2 - 2yz + 2xy + g(x,z)$$
$$\partial_z\varphi = 4x-2y-5z \implies \varphi = -\tfrac{5}{2}z^2 - 2yz + 4zx + h(x,y)$$

$$\implies \boxed{\varphi = x^2 + \tfrac{3}{2}y^2 - \tfrac{5}{2}z^2 - 2yz + 4zx + 2xy + \text{const.}}$$

[constant does not change $\nabla\varphi$]

For given $\underline{A}$,

$$\nabla\wedge\underline{A} = (0-(-4z-2y-2x))\underline{i} + (-2z+3y+2x)\underline{j} + ((4x-2z)-(2y+3z))\underline{k}$$
$$= (2x+2y+4z)\underline{i} + (2x+3y-2z)\underline{j} + (4x-2y-5z)\underline{k}$$
$$\boxed{= \underline{F}}$$

[Existence of $\underline{A}$ follows from $\nabla\cdot\underline{F} = 0$, and (5.6)].

2 From ES1, Q2 ii) $\nabla(\ln r) = \underline{r}/r^2$.

By (5.7) $\nabla^2(\ln r) = \nabla\cdot\nabla(\ln r) =$

$$= \partial_x(x/r^2) + \partial_y(y/r^2) + \partial_z(z/r^2)$$

$$= \left(\frac{1}{r^2} + x\left(\frac{-2}{r^3}\right)\cdot\frac{x}{r}\right) + \left(\frac{1}{r^2} + y\left(\frac{-2}{r^3}\right)\cdot\frac{y}{r}\right) + \left(\frac{1}{r^2} + z\left(\frac{-2}{r^3}\right)\frac{z}{r}\right)$$

$$\Rightarrow \nabla^2 \ln r = \frac{3}{r^2} - \frac{2(x^2+y^2+z^2)}{r^4} = \boxed{\frac{1}{r^2}}$$

3 $\underline{F} = (xy^2, yz^2, -x^3)$

i) $\nabla\cdot\underline{F} = y^2 + z^2 + 0 = \boxed{y^2+z^2}$

ii) $\nabla(\nabla\cdot\underline{F}) = (0, 2y, 2z)$ or $\boxed{2y\,\underline{j} + 2z\,\underline{k}}$.

iii) $\nabla\wedge\underline{F} = (0 - 2yz)\underline{i} + (0+3x^2)\underline{j} + (0 - 2xy)\underline{k}$

$$= \boxed{(-2yz, 3x^2, -2xy)}$$

iv) Using iii) $\nabla\wedge(\nabla\wedge\underline{F})$

$$= (-2x - 0)\underline{i} + (-2y + 2y)\underline{j} + (6x + 2z)\underline{k}$$

$$= \boxed{(-2x, 0, 6x + 2z)}$$

v) $\nabla^2\underline{F} = (\partial_{xx} + \partial_{yy} + \partial_{zz})(xy^2, yz^2, -x^3)$

$$= \boxed{(2x, 2y, -6x)}$$

vi) $(\underline{F}\cdot\nabla)\underline{F} = (xy^2\partial_x + yz^2\partial_y - x^3\partial_z)(xy^2, yz^2, -x^3)$

$$= \boxed{(xy^4 + 2xy^2z^2)\underline{i} + (yz^4 - 2x^3yz)\underline{j} + (-3x^3y^2)\underline{k}}$$

RHS of (1) $= (0, 2y, 2z) - (-2x, 0, 6x+2z) = (2x, 2y, -6x)$

$$= \nabla^2\underline{F} \quad ✓$$

RHS of (2) $= \nabla\left(\frac{1}{2}(x^2y^4 + y^2z^4 + x^6)\right)$

$$- (xy^2, yz^2, -x^3)\wedge(-2yz, 3x^2, -2xy)$$

$$= (xy^4 + 3x^5, 2x^2y^3 + yz^4, 2y^2z^3)$$

$$- (-2xy^2z^2 + 3x^5, 2x^3yz + 2x^2y^3, 3x^3y^2 + 2y^2z^3)$$

$$= (xy^4 + 2xy^2z^2, yz^4 - 2x^3yz, -3x^3y^2) = (\underline{F}\cdot\nabla)\underline{F} \quad ✓$$

4. $\varphi = x f(r) \Rightarrow \partial_x \varphi = f + x \frac{df}{dr} \frac{x}{r} = f + x^2 \frac{f'}{r}$

$$\frac{\partial^2 \varphi}{\partial x^2} = f' \frac{x}{r} + 2x \frac{f'}{r} + x^2 \frac{d}{dr}\left(\frac{f'}{r}\right) \cdot \frac{x}{r}.$$

$$= 3x \frac{f'}{r} + \frac{x^3}{r} \frac{d}{dr}\left(\frac{f'}{r}\right)$$

$$\partial_y \varphi = x \frac{df}{dr} \frac{y}{r} = xy \frac{f'}{r}.$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{x}{r} f' + xy \frac{d}{dr}\left(\frac{f'}{r}\right) \frac{y}{r} = \frac{x}{r} f' + \frac{xy^2}{r} \frac{d}{dr}\left(\frac{f'}{r}\right)$$

By symmetry $\partial_{zz} \varphi = \frac{x}{r} f' + \frac{xz^2}{r} \frac{d}{dr}\left(\frac{f'}{r}\right)$

$$\Rightarrow \nabla^2 \varphi = 5 \frac{x}{r} f' + \frac{x}{r}(x^2 + y^2 + z^2) \frac{d}{dr}\left(\frac{f'}{r}\right)$$

$$= x \left(\frac{5}{r} f' + r \frac{d}{dr}(f'/r)\right)$$

$$= x \left(\frac{4}{r} f' + f''\right)$$ as required.

$$\frac{1}{r^4} \frac{d}{dr}(r^4 f') = 0 \Rightarrow \frac{d}{dr}(r^4 f') = 0$$

$$\Rightarrow r^4 f' = \text{const} \Rightarrow f' = C/r^4$$

$$\Rightarrow \boxed{f = A/r^3 + B}$$

A, B constants.

5 i) $b_{jj} = b_{11} + b_{22} + b_{33} = 1 + 3 + 4 = \boxed{8}$

ii) $b_{i1} b_{i2} = b_{11} b_{12} + b_{21} b_{22} + b_{31} b_{32} = -1 - 6 + 0 = \boxed{-7}$

iii) $b_{3p} b_{2p} = b_{31} b_{21} + b_{32} b_{22} + b_{33} b_{23} = -4 + 0 + 4 = \boxed{0}$

iv) $\delta_{1i} b_{1i} = \delta_{11} b_{11} + \delta_{12} b_{12} + \delta_{13} b_{13}$

$= b_{11}$ SEE SUBSTITUTION PROPERTY, (5.11)

v) $\underbrace{\delta_{jk} b_{2j}}_{j \to k} b_{3k} = b_{2k} b_{3k} = \boxed{0}$ BY iii)

vi) $\varepsilon_{1jk} b_{jk} = \varepsilon_{123} b_{23} + \varepsilon_{132} b_{32} = 1 - 0 = \boxed{1}$

6 i) $\underbrace{\delta_{ij} \delta_{ij}}_{j \to i} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$.

USING THE SUBSTITUTION PROPERTY (5.11)

ii) $\underbrace{\delta_{ij} F_{ik}}_{i \to j} = F_{jk}$.

7 $\{\nabla \wedge (\underline{u} \wedge \underline{v})\}_i = \varepsilon_{ijk} \partial_j (\underline{u} \wedge \underline{v})_k$.

$(\underline{u} \wedge \underline{v})_k = \varepsilon_{klm} u_l v_m$, AVOIDING j, k.

$\Rightarrow \{\nabla \wedge (\underline{u} \wedge \underline{v})\}_i = \varepsilon_{ijk} \varepsilon_{klm} \partial_j (u_l v_m)$ AS REQUIRED.

$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j (u_l v_m)$ BY (5.12)

$l \to i \quad m \to j \quad m \to i \quad l \to j$

$= \partial_j (u_i v_j) - \partial_j (u_j v_i)$ (*)

$= u_i \partial_j v_j + v_j \partial_j u_i - u_j \partial_j v_i - v_i \partial_j u_j$

$= u_i (\nabla \cdot \underline{v}) + (\underline{v} \cdot \nabla) u_i - (\underline{u} \cdot \nabla) v_i - v_i (\nabla \cdot \underline{u})$

$\Rightarrow \nabla \wedge (\underline{u} \wedge \underline{v}) = (\underline{v} \cdot \nabla) \underline{u} - (\underline{u} \cdot \nabla) \underline{v} + \underline{u} (\nabla \cdot \underline{v}) - \underline{v} (\nabla \cdot \underline{u})$

8 $\{\nabla \wedge (\nabla \wedge \underline{F})\}_i$ AS Q7 UP TO (*), WITH $u \to \partial$, $v \to F$.

$= \partial_j \partial_i F_j - \partial_j \partial_j F_i$

$= \partial_i \partial_j F_j - \partial_j \partial_j F_i = \nabla_i (\nabla \cdot \underline{F}) - \nabla^2 F_i$