1. Using CPs (r, θ, z), prove that $V = r^n \cos n\theta$, where n is a constant, satisfies $\nabla^2 V = 0$.

   Show also that if $W = W(r)$ and $\nabla^2 W = 0$, then $W = A + B \ln r$, where A and B are constants.

2. Evaluate the surface integral, $\int \boldsymbol{F} \cdot \boldsymbol{dS}$, for the surface of the cylindrical region $r \leq a$, $0 \leq z \leq h$, for the case $\boldsymbol{F} = \boldsymbol{r}$. Note that in cylindrical polars, $\boldsymbol{r} = r\hat{\boldsymbol{r}} + z\hat{\boldsymbol{z}}$.

3. Using appropriate coordinates, verify the divergence theorem for:

   (i) $\boldsymbol{F} = r^2 \boldsymbol{r}$ for the spherical region $r \leq a$, where $\boldsymbol{r} = (x, y, z)$ and $r = |\boldsymbol{r}|$;

   (ii) $\boldsymbol{F} = r^2 \hat{\boldsymbol{r}} + Az^2 \hat{\boldsymbol{z}}$, where A is a constant and (r, θ, z) are cylindrical polars, for the cylindrical region $r \leq a,\ 0 \leq z \leq h$.

4. By applying Stokes's Theorem to $\phi \boldsymbol{a}$, where $\boldsymbol{a}$ is a constant vector, use the expansion $\nabla \times (\phi \mathbf{v}) = \phi \nabla \times \mathbf{v} + \nabla \phi \times \mathbf{v}$ to show that

$$\oint_C \phi \, \boldsymbol{dl} = -\int_S \nabla \phi \times \boldsymbol{dS},$$

   where S is an open surface bounded by a closed curve C.

   Verify this result for the case $\phi = 3a^2 y + x^2 z$, where a is a constant and C is the circle $x^2 + y^2 = a^2$.

5. *Flow through a pipe.* Provided that the flow rate is not too high, flow through a cylindrical pipe of radius a is given by $\boldsymbol{u} = 2U(1 - \frac{r^2}{a^2})\hat{\boldsymbol{z}}$ (Hagen–Poiseuille flow), where U is a constant. Show that $\nabla \cdot \boldsymbol{u} = 0$. Show that the mean flow speed is U, i.e. that the average speed of the fluid passing through a cross-section of the pipe is U.

6. *Flow around a cylinder.* Using CPs, find $\boldsymbol{u} = \nabla \times (\psi \hat{\boldsymbol{z}})$ for the flow of Worked Example 8.3 with

$$\psi = U \left(r - \frac{a^2}{r} \right) \sin \theta \, .$$

   Show that the flow is irrotational, i.e. $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \mathbf{0}$. It follows that $\boldsymbol{u} = \nabla \phi$ for some velocity potential ϕ. Determine ϕ for this case.