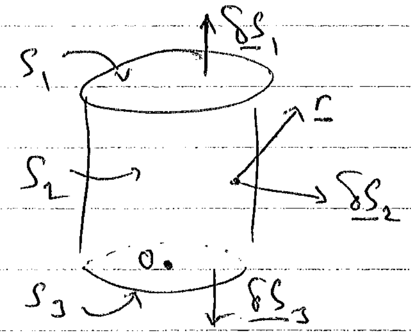
MAS280 Example Sheet 5

1 CPs → FORMULA SHEET (CP.4) $V = r^n \cos n\theta$

$$\nabla^2 V = \frac{1}{r}\partial_r(r\partial_r V) + \frac{1}{r^2}\partial_{\theta\theta}V + 0$$
$$= \frac{\cos n\theta}{r} \cdot \frac{d}{dr}(nr^n) - \frac{n^2}{r^2} r^n \cos n\theta$$
$$= n^2 \cdot r^{n-2}(\cos n\theta - \cos n\theta) = \boxed{0}$$

$$\nabla^2 W = 0, \quad W = W(r) \Rightarrow \frac{1}{r}\partial_r(r\partial_r W) = 0$$
$$\Rightarrow \partial_r(r\partial_r W) = 0 \Rightarrow r\partial_r W = B$$
$$\Rightarrow \partial_r W = B/r \Rightarrow \boxed{W = B\ln r + A}.$$

2 

CPs: $\underline{r} = r\hat{\underline{r}} + z\hat{\underline{z}}$

$$\delta\underline{S}_1 = r\delta r\delta\theta\,\hat{\underline{z}}$$
$$\delta\underline{S}_2 = r\delta\theta\,\delta z\,\hat{\underline{r}}$$
$$\delta\underline{S}_3 = -r\delta r\delta\theta\,\hat{\underline{z}} = -\delta\underline{S}_1$$

$\underline{r}\cdot\delta\underline{S}_1 = \underline{r}\cdot\hat{\underline{z}}\, r\delta r\delta\theta$, $\underline{r}\cdot\hat{\underline{z}} = z = h$ on S_1.

$\underline{r}\cdot\delta\underline{S}_2 = \underline{r}\cdot\hat{\underline{r}}\, r\delta\theta\delta z$, $\underline{r}\cdot\hat{\underline{r}} = r = a$ on S_2.

$\underline{r}\cdot\delta\underline{S}_3 = -\underline{r}\cdot\hat{\underline{z}}\, r\delta r\delta\theta$, $\underline{r}\cdot\hat{\underline{z}} = z = 0$ on S_3.

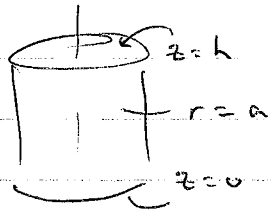
$$\int_S \underline{r}\cdot d\underline{S} = \int_{S_1}\underline{r}\cdot d\underline{S}_1 + \int_{S_2}\underline{r}\cdot d\underline{S}_2 + \int_{S_3}\underline{r}\cdot d\underline{S}_3$$
$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} hr\,dr\,d\theta + \int_{z=0}^{h}\int_{\theta=0}^{2\pi} a^2\,d\theta\,dz + 0$$
$$= 2\pi\cdot h\cdot\tfrac{1}{2}a^2 + 2\pi\cdot h\cdot a^2 = \boxed{3\pi h a^2}.$$

3 i) ON $r=a$, $\underline{F} = a^3\hat{\underline{r}}$ AND $d\underline{S} = a^2 \sin\theta \, d\theta \, d\varphi \, \hat{\underline{r}}$ IN SPS

$$\Rightarrow \int \underline{F} \cdot d\underline{S} = 2\pi a^5 \int_0^\pi \sin\theta \, d\theta = \boxed{4\pi a^5}$$

(SP.4) FORMULA SHEET $\Rightarrow \quad \nabla \cdot \underline{F} = \frac{1}{r^2}\frac{d}{dr}(r^5) = 5r^2$

$$\int \nabla \cdot \underline{F} \, dV = 5\int_0^{2\pi}\int_0^\pi\int_0^a r^2 \cdot r^2 \sin\theta \, dr \, d\theta \, d\varphi$$
$$= 5 \cdot 2\pi \cdot 2 \cdot \frac{1}{5} a^5 = \boxed{4\pi a^5}$$

ii) 

$\underline{F} = r^2\hat{\underline{r}} + Az^2\hat{\underline{z}}$

- ON $z=0$, $d\underline{S} = -r\,dr\,d\theta\,\underline{k}$
  $\Rightarrow \underline{F} \cdot d\underline{S} = 0$ AS $z=0$
- ON $z=h$, $d\underline{S} = r\,dr\,d\theta\,\underline{k}$
  $\Rightarrow \underline{F} \cdot d\underline{S} = A h^2 r \, dr \, d\theta$
- ON $r=a$, $d\underline{S} = a\,d\theta\,dz\,\hat{\underline{r}} \Rightarrow \underline{F} \cdot d\underline{S} = a^3 \, d\theta \, dz$

$$\Rightarrow \int \underline{F} \cdot d\underline{S} = 0 + Ah^2 \cdot \pi a^2 + a^3 \cdot 2\pi \cdot h = \boxed{\pi a^2 h(2a + Ah)}$$

BY (CP.2) $\nabla \cdot \underline{F} = \frac{1}{r}\frac{d}{dr}(r^3) + \frac{d}{dz}(Az^2) = 3r + 2Az$

$$\Rightarrow \int \nabla \cdot \underline{F} \, dV = \iiint (3r + 2Az) \, r \, dr \, d\theta \, dz$$
$$= 2\pi h \int_0^a 3r^2 \, dr + 4\pi A \int_0^h z \, dz \cdot \int_0^a r \, dr$$
$$= 2\pi h a^3 + 4\pi A^2 \cdot \frac{1}{2}h^2 \cdot \frac{1}{2}a^2 = \boxed{\pi a^2 h(2a + Ah)}$$

4 $\nabla \wedge (\phi \underline{a}) = \varphi \underbrace{\nabla \wedge \underline{a}}_{=0 \text{ AS } \underline{a} \text{ CONST,}} + (\nabla \varphi) \wedge \underline{a} = (\nabla \varphi) \wedge \underline{a} \quad (*)$

STOKES' THM

$$\Rightarrow \oint_C \varphi \underline{a} \cdot d\underline{l} = \int \nabla \wedge (\varphi \underline{a}) \cdot d\underline{S}$$

$$= \int (\nabla \varphi) \wedge \underline{a} \cdot d\underline{S} \quad \text{BY } (*)$$

$$= -\int \underline{a} \cdot \nabla \varphi \wedge d\underline{S}$$

$\underline{a}$ CONST $\Rightarrow$ $\underline{a} \cdot \oint_C \varphi \, d\underline{l} = -\underline{a} \cdot \int_S \nabla \varphi \wedge d\underline{S}$

$\underline{a}$ ARBITRARY $\Rightarrow$ $\oint_C \varphi \, d\underline{l} = -\int_S \nabla \varphi \wedge d\underline{S}$

AS REQUIRED.

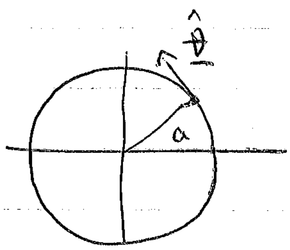
$\varphi = 3a^2 y + x^2 z, \quad C : x^2 + y^2 = a.$

ON C, $d\underline{l} = a \, d\theta \, \hat{\underline{\theta}}$

$= a(-\sin\theta \, \underline{i} + \cos\theta \, \underline{j}) \, d\theta$

AND $\varphi = 3a^2 \cdot a \sin\theta + 0 \leftarrow z = 0$

$= 3a^3 \sin\theta$



$\hat{\underline{\theta}} = -\sin\theta \, \underline{i} + \cos\theta \, \underline{j}$

$$\Rightarrow \oint_C \varphi \, d\underline{l} = -3a^4 \underbrace{\int_0^{2\pi} \sin^2\theta \, d\theta}_{=\pi} \, \underline{i} + 3a^4 \underbrace{\int_0^{2\pi} \sin\theta \cdot \cos\theta \, d\theta}_{=0,} \, \underline{j}$$

$$= \boxed{-3\pi a^4 \underline{i}}$$

ALSO $\nabla \varphi = (2xz, 3a^2, x^2)$, $\quad d\underline{S} = r \, dr \, d\theta \, \underline{k}$

$$-\nabla \varphi \wedge d\underline{S} = -(2xz, 3a^2, x^2) \wedge (0, 0, r \, dr \, d\theta)$$

$$= (-3a^2 r \, dr \, d\theta, \underbrace{2xz r \, dr \, d\theta}_{=0 \text{ ON } S.}, 0)$$

$$\Rightarrow -\int \nabla \varphi \wedge d\underline{S} = -3a^2 \int r \, dr \, d\theta \, \underline{i} = -3a^2 \cdot \pi a^2 \, \underline{i}$$

$$= \boxed{-3\pi a^4 \underline{i}}$$

5 GEOMETRY $\Rightarrow$ CPs. $\quad u_1 = u_2 = 0$, $u_3 = 2U\left(1 - \frac{r^2}{a^2}\right)$

FORMULA SHEET (CP.2) $\Rightarrow \nabla \cdot \underline{u} = 0 + 0 + \partial_z u_3 = \boxed{0.}$

MEAN OF z-COMPONENT OF $\underline{u}$ OVER CROSS-SECTION δS

$$\bar{u} = \frac{1}{\pi a^2} \int \underline{u} \cdot \hat{\underline{z}}\, dS = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a 2U\left(1 - \frac{r^2}{a^2}\right) r\, dr\, d\theta$$

$$= \frac{1}{\pi a^2} \cdot 2\pi \cdot 2U \underbrace{\left[\frac{1}{2} r^2 - \frac{1}{4} \frac{r^4}{a^2}\right]_0^a}_{\frac{1}{4} a^2} = \boxed{U.}$$

6. $\underline{u} = \nabla \wedge (\psi \hat{\underline{z}})$ (CP.3) WITH $F_1 = F_2 = 0$, $F_3 = \psi = U\left(r - \frac{a^2}{r^2}\right)\sin\theta$

$$\underline{u} = \left(\frac{1}{r}\partial_\theta \psi\right)\hat{\underline{r}} - \left(\partial_r \psi\right)\hat{\underline{\theta}}$$
$$= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\, \hat{\underline{r}} - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\, \hat{\underline{\theta}}$$

$\underline{\omega} = \nabla \wedge \underline{u}$ (CP.3) F_1, F_2, $F_3 = 0$,

$$= 0\,\hat{\underline{r}} + 0\,\hat{\underline{\theta}} + \left[\frac{1}{r}\partial_r\left(rU\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right) + \frac{1}{r}\partial_\theta U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\right]\hat{\underline{z}}$$

$$= \left[\frac{U\sin\theta}{r}\left(1 - \frac{a^2}{r^2}\right) - \frac{U\sin\theta}{r}\left(1 - \frac{a^2}{r^2}\right)\right]\hat{\underline{z}} = \boxed{\underline{0}}$$

$\Rightarrow$ IRROTATIONAL.

POTENTIAL $\Rightarrow$

$\underline{u} = \nabla \varphi$

(CP.1) $\Rightarrow \partial_r \varphi = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \Rightarrow \varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta + f(\theta, z)$

$\Rightarrow \frac{1}{r}\partial_\theta \varphi = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$

$\Rightarrow \partial_\theta \varphi = -rU\left(1 + \frac{a^2}{r^2}\right)\sin\theta$

$\Rightarrow \varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta + g(r, z)$

$\Rightarrow$ TAKE $\boxed{\varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta}$.

(SEE SECTIONS 8.3 & 10.3)