

1. Find the vorticity $\underline{\omega}$, the acceleration $\frac{D\underline{u}}{Dt}$, and the deviatoric stress tensor d_{ij} for each of the following steady flows. ["steady" $\Rightarrow \frac{\partial}{\partial t} \equiv 0$]

(a) $\underline{u}(\underline{x}) = U \left\{ 1 - \frac{x_1^2 + x_2^2}{a^2} \right\} \underline{e}_3$; [U, a constant. (1.17) in Notes]

(b) $\underline{u}(\underline{x}) = \underline{\Omega} \times \underline{x}$; [$\underline{\Omega}$ constant. Solid-body rotation]

2. Given that $\nabla \cdot \underline{u} = 0$, show that the stress tensor \underline{t} on a surface element with outward normal \underline{n} is given by

$$\underline{t} = -p \underline{n} + 2\mu (\underline{n} \cdot \nabla) \underline{u} + \mu \underline{n} \times \underline{\omega}.$$

[HINT. Use (1.24) to expand the last term on the R.H.S.]

3. The vertices A, B, C, D of a tetrahedron have coordinates $(7\delta, 0, 0)$, $(0, 7\delta, 0)$, $(0, 0, -4\delta)$, $(0, 0, 7\delta)$ respectively, where δ is so small that the stress tensor σ_{ij} can be assumed to be constant over the tetrahedron.

The stress vectors outwards from the faces CDA, DAB, ABC are $(2, 2, 1) \text{ Nm}^{-2}$, $\frac{1}{\sqrt{3}}(1, -5, -1) \text{ Nm}^{-2}$, $(-2, -1, 2) \text{ Nm}^{-2}$ respectively.

- (a) Verify that the planes CDA, DAB, ABC have equations $x_2 = 0$, $x_1 + x_2 + x_3 = 7\delta$, $4x_1 + 4x_2 - 7x_3 = 28\delta$ respectively. Hence determine the outward normals to these faces of the tetrahedron, and show that

$$\sigma_{ij} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & -1 \\ 2 & -1 & -2 \end{pmatrix} \text{ Nm}^{-2}.$$

- (b) You are given that the faces BCD, CDA, DAB, ABC have areas $\frac{77}{2} \delta^2$, $\frac{77}{2} \delta^2$, $\frac{49\sqrt{3}}{2} \delta^2$, $\frac{63}{2} \delta^2$ respectively. Show that the resultant force on the tetrahedron due to this stress tensor is zero.

4. The stress tensor at a point O in a fluid has components

$$\frac{1}{4} \begin{pmatrix} 1 & 0 & 3\sqrt{3} \\ 0 & -8 & 0 \\ 3\sqrt{3} & 0 & -5 \end{pmatrix} \text{ Nm}^{-2}$$

- (a) Find the principal stresses, showing that two are equal. What are the principal axes of stress?

- (b) Find the stress vector \underline{t} on a surface element with normal $\underline{n} = (n_1, n_2, n_3)$

