

1. Find the vorticity  $\underline{\omega}$ , the acceleration  $\frac{D\underline{u}}{Dt}$ , and the deviatoric stress tensor  $d_{ij}$  for each of the following steady flows. ["steady"  $\Rightarrow \frac{\partial}{\partial t} \equiv 0$ ]

(a)  $\underline{u}(\underline{x}) = U \left\{ 1 - \frac{x_1^2 + x_2^2}{a^2} \right\} \underline{e}_3$ ; [ $U, a$  constant. (1.17) in Notes]

(b)  $\underline{u}(\underline{x}) = \underline{\Omega} \times \underline{x}$ ; [ $\underline{\Omega}$  constant. Solid-body rotation]

2. Given that  $\nabla \cdot \underline{u} = 0$ , show that the stress tensor  $\underline{t}$  on a surface element with outward normal  $\underline{n}$  is given by

$$\underline{t} = -p\underline{n} + 2\mu(\underline{n} \cdot \nabla)\underline{u} + \mu\underline{n} \times \underline{\omega}.$$

[HINT. Use (1.24) to expand the last term on the R.H.S.]

3. The vertices  $A, B, C, D$  of a tetrahedron have coordinates  $(7\delta, 0, 0)$ ,  $(0, 7\delta, 0)$ ,  $(0, 0, -4\delta)$ ,  $(0, 0, 7\delta)$  respectively, where  $\delta$  is so small that the stress tensor  $\sigma_{ij}$  can be assumed to be constant over the tetrahedron. The stress vectors outwards from the faces  $CDA, DAB, ABC$  are  $(2, 2, 1) \text{ Nm}^{-2}$ ,  $\frac{1}{\sqrt{3}}(1, -5, -1) \text{ Nm}^{-2}$ ,  $(-2, -1, 2) \text{ Nm}^{-2}$  respectively.

- (a) Verify that the planes  $CDA, DAB, ABC$  have equations  $x_2 = 0$ ,  $x_1 + x_2 + x_3 = 7\delta$ ,  $4x_1 + 4x_2 - 7x_3 = 28\delta$  respectively. Hence determine the outward normals to these faces of the tetrahedron, and show that

$$\sigma_{ij} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & -1 \\ 2 & -1 & -2 \end{pmatrix} \text{ Nm}^{-2}.$$

- (b) You are given that the faces  $BCD, CDA, DAB, ABC$  have areas  $\frac{77}{2}\delta^2$ ,  $\frac{77}{2}\delta^2$ ,  $\frac{49\sqrt{3}}{2}\delta^2$ ,  $\frac{63}{2}\delta^2$  respectively. Show that the resultant force on the tetrahedron due to this stress tensor is zero.

4. The stress tensor at a point  $O$  in a fluid has components

$$\frac{1}{4} \begin{pmatrix} 1 & 0 & 3\sqrt{3} \\ 0 & -8 & 0 \\ 3\sqrt{3} & 0 & -5 \end{pmatrix} \text{ Nm}^{-2}$$

- (a) Find the principal stresses, showing that two are equal. What are the principal axes of stress?

- (b) Find the stress vector  $\underline{t}$  on a surface element with normal  $\underline{n} = (n_1, n_2, n_3)$

