

1. Viscous incompressible fluid flows in the direction of Oz through a straight pipe. Show that all the governing equations are satisfied by $P = p_0 - Gz$, $\underline{u} = (0, 0, w(x, y))$, where p_0 and G are constants, provided

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu}. \quad (*)$$

Suppose the cross-section is the equilateral triangle of side $2a$ bounded by $y = 0, (y - \sqrt{3}a) = \pm \sqrt{3}x$. Show that $(*)$ and the no slip condition are everywhere satisfied by

$$w = \alpha y \{ (y - \sqrt{3}a)^2 - 3x^2 \},$$

for one value of the constant α . Deduce that the maximum value of w is $G a^2 / (9\mu)$, and that the mean velocity is $G a^2 / (20\mu)$.

2. Viscous incompressible fluid occupies the region between the two infinite flat rigid plates $y = 0$ and $y = d$. Initially the fluid and the plates are at rest. At $t = 0$, the plate at $y = d$ is jerked into motion at constant speed W in the direction of Oz . The fluid velocity $\underline{u} = (0, 0, w(y, t))$, and there is no pressure gradient.

(i) Show that $\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial y^2}$ with $w(0, t) = 0$ ($t > 0$), $w(d, t) = W$ ($t > 0$), $w(y, 0) = 0$ ($0 < y < d$).

(ii) What is $\lim_{t \rightarrow \infty} w(y, t)$? (HINT. See Example 8 in the Notes.)

(iii) Write $w(y, t) = Wy/d - w_*(y, t)$. Show that $\frac{\partial w_*}{\partial t} = \nu \frac{\partial^2 w_*}{\partial y^2}$, and write down the boundary and initial conditions satisfied by w_* .

(iv) Use the method of separation of variables and the principle of superposition (Yr 2 DEs course) to show that there are constants α_n ($n = 1, 2, \dots$) such that

$$w_* = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi y}{d}\right) e^{-n^2 \pi^2 \nu t / d^2}$$

(v) Use Fourier series to determine the α_n .

(vi) Hence show that the volumetric rate of flow per unit width in the direction of Ox is

$$\int_0^d w dy = \frac{Wd}{2} \left\{ 1 - \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{\exp[-(2m+1)^2 \pi^2 \nu t / d^2]}{(2m+1)^2} \right\}.$$

