

1. Viscous incompressible fluid flows in the direction of  $Oz$  through a straight pipe. Show that all the governing equations are satisfied by  $P = p_0 - Gz$ ,  $\underline{u} = (0, 0, w(x, y))$ , where  $p_0$  and  $G$  are constants, provided

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{G}{\mu}. \quad (*)$$

Suppose the cross-section is the equilateral triangle of side  $2a$  bounded by  $y = 0$ ,  $(y - \sqrt{3}a) = \pm \sqrt{3}x$ . Show that  $(*)$  and the no slip condition are everywhere satisfied by

$$w = \alpha y \{ (y - \sqrt{3}a)^2 - 3x^2 \},$$

for one value of the constant  $\alpha$ . Deduce that the maximum value of  $w$  is  $G a^2 / (9\mu)$ , and that the mean velocity is  $G a^2 / (20\mu)$ .

2. Viscous incompressible fluid occupies the region between the two infinite flat rigid plates  $y = 0$  and  $y = d$ . Initially the fluid and the plates are at rest. At  $t = 0$ , the plate at  $y = d$  is jerked into motion at constant speed  $W$  in the direction of  $Oz$ . The fluid velocity  $\underline{u} = (0, 0, w(y, t))$ , and there is no pressure gradient.

(i) Show that  $\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial y^2}$  with  $w(0, t) = 0$  ( $t > 0$ ),  $w(d, t) = W$  ( $t > 0$ ),  $w(y, 0) = 0$  ( $0 < y < d$ ).

(ii) What is  $\lim_{t \rightarrow \infty} w(y, t)$ ? (HINT. See Example 8 in the Notes.)

(iii) Write  $w(y, t) = Wy/d - w_*(y, t)$ . Show that  $\frac{\partial w_*}{\partial t} = \nu \frac{\partial^2 w_*}{\partial y^2}$ , and write down the boundary and initial conditions satisfied by  $w_*$ .

(iv) Use the method of separation of variables and the principle of superposition (Yr 2 DEs course) to show that there are constants  $\alpha_n$  ( $n = 1, 2, \dots$ ) such that

$$w_* = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi y}{d}\right) e^{-n^2 \pi^2 \nu t / d^2}$$

(v) Use Fourier series to determine the  $\alpha_n$ .

(vi) Hence show that the volumetric rate of flow per unit width in the direction of  $Ox$  is

$$\int_0^d w dy = \frac{Wd}{2} \left\{ 1 - \frac{8}{\pi^2} \sum_{m=0}^{\infty} \frac{\exp[-(2m+1)^2 \pi^2 \nu t / d^2]}{(2m+1)^2} \right\}.$$

