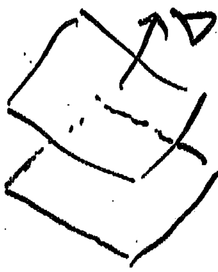
W. Ex. 1.2 (A)

$$\phi = 2x - 3y + z$$

$$\frac{\partial \phi}{\partial x} = 2, \quad \frac{\partial \phi}{\partial y} = -3, \quad \frac{\partial \phi}{\partial z} = 1$$

$$\nabla \phi = 2\underline{i} - 3\underline{j} + \underline{k}$$



$\nabla\phi$

$\phi = c_2$ Parallel planes

$\rightarrow \nabla\phi$ const.

$\phi = c_1$

W. Ex. 1.2 (B)

$$\phi = r^n, \quad r = (x^2+y^2+z^2)^{\frac{1}{2}}.$$

$$\frac{\partial r}{\partial x} = 2x \cdot \tfrac{1}{2} \cdot (x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{x}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r} \cdot \frac{\partial r}{\partial x} = n r^{n-1} \cdot \frac{x}{r}.$$

$$\Rightarrow \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \nabla \phi = n r^{n-1} \frac{(x, y, z)}{r}$$

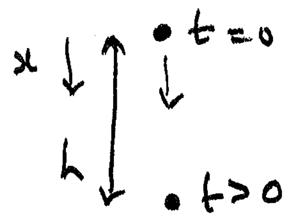
i.e. $\nabla \phi = n r^{n-1} \underline{r}/r = n r^{n-1} \hat{\underline{r}}$.

$\phi = \text{const} \Rightarrow r = \text{const}$

$\nabla\phi \propto \underline{r}$ Concentric spheres.

W. Ex 1.3

(A)



$F = ma$

$mg = ma$

$mgt = mv + \cancel{C}_{\to 0}$ $\quad v = 0$ at $t = 0$

$\frac{1}{2}mgt^2 = mx + \cancel{D}_{\to 0}$ $\quad x = 0$ at $t = 0$

$\Rightarrow v = gt$ and $x = h$ when $\frac{1}{2}gt^2 = h$

$\Rightarrow t = \sqrt{2h/g}$

$\Rightarrow v = g \cdot \sqrt{2h/g} = \sqrt{2gh}$

(B)

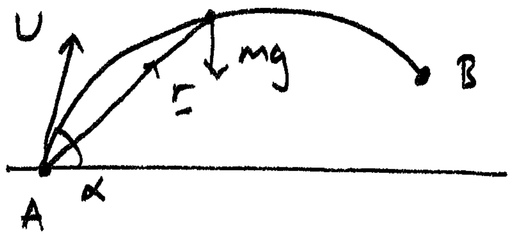
$V = mgh$, $E = 0$

$V + E = \text{const}$

$V = 0 \Rightarrow E = mgh$. Also $E = \frac{1}{2}mv^2$.

$\Rightarrow gh = \frac{1}{2}v^2$, $v = \sqrt{2gh}$.

W. Ex. 2.1

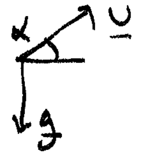


$t_A = 0, \quad t_B = \tau.$

$$E(t=0) = \tfrac{1}{2} m U^2, \qquad E(t) = \tfrac{1}{2} m \underline{\dot{r}} \cdot \underline{\dot{r}}$$

$$\underline{F} = m\underline{a} \Rightarrow \quad m\underline{\ddot{r}} = m\underline{g} \quad \Rightarrow \underline{\ddot{r}} = \underline{g} \quad \Rightarrow \underline{\dot{r}} = \underline{g}t + \underline{U}.$$

Now,
$$E(\tau) = \tfrac{1}{2} m (\underline{g}\tau + \underline{U}) \cdot (\underline{g}\tau + \underline{U}) = \tfrac{1}{2} m (g^2\tau^2 + 2\tau \underline{g} \cdot \underline{U} + U^2)$$



$$\underline{g} \cdot \underline{U} = -gU\sin\alpha$$

$\Rightarrow$ Change in K.E.
$$E(\tau) - E(0) = \tfrac{1}{2} m (g^2\tau^2 - 2\tau g U \sin\alpha).$$

Work Done (2.4)
$$W = \int_0^\tau \underline{F} \cdot \underline{\dot{r}} \, dt$$

$$\underline{F} \cdot \underline{\dot{r}} = m\underline{g} \cdot (\underline{g}t + \underline{U}) = mg^2 t - mgU\sin\alpha$$

$$\Rightarrow W = \left[\tfrac{1}{2} m g^2 t^2 - mgU\sin\alpha \cdot t \right]_0^\tau = E(\tau) - E(0)$$

Work-Energy principle holds (2.5)

W. Ex. 2.2 (A)

$$\underline{F} = -\nabla V, \qquad V = -\frac{GMm}{r}$$

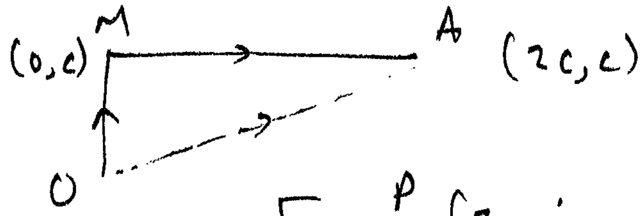
$$\underline{F} = GMm\,\nabla\left(\frac{1}{r}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\cdot\frac{\partial r}{\partial x}$$ ← see W. Ex. 1.2 (B)

$$= -\frac{1}{r^2}\cdot\frac{x}{r}$$

$$\underline{F} = -\frac{GMm}{r^2}\cdot\frac{(x,y,z)}{r} = -\frac{GMm}{r^2}\,\hat{\underline{r}}$$

W. Ex 2.2 (B)



$$\underline{F} = \frac{P}{c}(3x\,\underline{i} + y\,\underline{j})$$

$$\delta W = \underline{F}\cdot\underline{\delta r} = \frac{P}{c}(3x\,\delta x + y\,\delta y)$$

OM: $\delta x = 0 \Rightarrow \int_{OM} dW = \frac{P}{c}\int_0^c y\,dy = \frac{P}{c}\cdot\frac{1}{2}c^2 = \frac{1}{2}Pc$.

MA: $\delta y = 0 \Rightarrow \int_{MA} dW = \frac{P}{c}\int_0^{2c} 3x\,dx = \frac{P}{c}\cdot\frac{3}{2}(2c)^2 = 6Pc$

$$\Rightarrow W_1 = \frac{13}{2}Pc.$$

OA: $\int_{OA} dW = \frac{P}{c}\left\{\int_0^{2c} 3x\,dx + \int_0^c y\,dy\right\}$ (as above!)

$$= \frac{13}{2}Pc.$$

IF $\underline{F} = -\nabla V$, then

$-\partial_x V = \frac{3P}{c}x \quad \Rightarrow -V = \frac{3P}{2c}x^2 + f(y,z)$

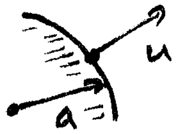
$-\partial_y V = \frac{P}{c}y \quad \Rightarrow -V = \frac{P}{2c}y^2 + g(x,z)$

$-\partial_z V = 0 \quad \Rightarrow -V = h(x,y)$

Together, $-V = \frac{P}{2c}(3x^2+y^2) + V_0$

CONST.

W. Ex 2.2 (c)



$E + V = \text{const}$ and

$V = -\dfrac{GMm}{r}$ from W. Ex. 2.2 (A)

$$\Rightarrow \underset{(\text{INITIALLY})}{\tfrac{1}{2}mu^2 - \dfrac{GMm}{a}} = \underset{(\text{LATER})}{\tfrac{1}{2}mv^2 - \dfrac{GMm}{r}}$$

$$\Rightarrow v^2 = u^2 - \frac{2GM}{a} + \frac{2GM}{r}$$

Let $U_e = \left(\dfrac{2GM}{a}\right)^{\frac{1}{2}}$,

then $v^2 = u^2 - U_e^2 + \dfrac{2GM}{r}$

i) If $u > U_e$, then $v^2 > 0 \;\forall r. \Rightarrow$ escape

ii) If $u < U_e$, then reaches max height when $v = 0$, at

$$r = \frac{2GM}{U_e^2 - u^2} = \frac{U_e^2 a}{U_e^2 - u^2} = \frac{a}{1 - u^2/U_e^2}$$

Observe,

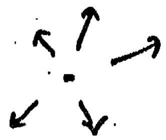
as $r \to \infty$, $E_\infty + V_\infty = \text{const} \Rightarrow E_\infty = \text{const}$.

$v \to 0$ as $r \to \infty$ if total energy $= 0$

W. Ex 5.1 (A)

$$\underline{F} = \underline{r} = (x, y, z)$$

$$\nabla\cdot\underline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(x, y, z) = 1+1+1 = 3$$



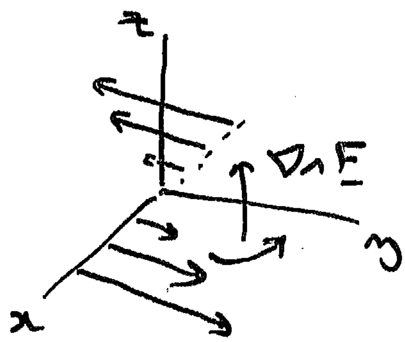
$$\nabla\cdot\underline{F} > 0$$

$$\nabla\wedge\underline{F} = \left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y, \frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z, \frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = (0, 0, 0) = \underline{0}$$

No rotation

(B)

$$\underline{F} = x\underline{j}, \quad F_2 = x, \; F_1 = F_3 = 0.$$



$$\nabla\wedge\underline{F} = \left(-\frac{\partial}{\partial z}x, 0, \frac{\partial}{\partial x}x\right)$$

$$= \underline{k}$$

W, Ex 5.3

Let $f' = \dfrac{df}{dr}$. W. Ex 1.2(B): $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

Now $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x} = f' \cdot \dfrac{x}{r}$

$$\partial_{xx} f = \frac{f'}{r} + x\,\partial_x\left(\frac{f'}{r}\right) = \frac{f'}{r} + x\frac{\partial}{\partial r}\left(\frac{f'}{r}\right)\cdot\frac{\partial r}{\partial x}$$

$$= \frac{f'}{r} + \frac{x^2}{r}\left(\frac{f''}{r} - \frac{f'}{r^2}\right)$$

$$\Rightarrow \nabla^2 f = 3\frac{f'}{r} + \frac{r^2}{r}\left(\frac{f''}{r} - \frac{f'}{r^2}\right) = \frac{2f'}{r} + f'' = \frac{1}{r^2}\frac{d}{dr}\left(r^2 f'\right)$$

as required

Now $\nabla^2 f = 0 \Rightarrow$

$r^2 f' = \text{const}$, $f' = C/r^2$

so $f = \dfrac{A}{r} + B$ is the general solution for Laplace's eqn.

W. Ex. 5.4

$$\underline{u} = \frac{U}{L}(x, -y, 0)$$

$$\underline{u}\cdot\nabla = \frac{U}{L}(x\partial_x - y\partial_y + 0)$$

$$\underline{u}\cdot\nabla\underline{u} = \frac{U^2}{L^2}(x\partial_x - y\partial_y)(x\underline{i} - y\underline{j})$$

$$= \frac{U^2}{L^2}(x\partial_x x)\underline{i} + \frac{U^2}{L^2}(-y\partial_y(-y))\underline{j}$$

$$= \frac{U^2}{L^2}(x\underline{i} + y\underline{j})$$

$$= \frac{U^2}{L^2}(x, y, 0).$$

Ex. 5.5.1

a) $\delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

b) $\epsilon_{ijk}\epsilon_{ijk} = \sum_{ijk} \epsilon_{ijk}^2 = 6$ (the number of non-zero ϵ_{ijk})

c) $\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$
$= a_ib_i$

d) $\underline{r} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3$
$= x_i\underline{e}_i$

e) $$\underline{a}\wedge\underline{b} = \underbrace{(a_2b_3 - a_3b_2)}_{\epsilon_{123}\cdot a_2b_3 + \epsilon_{132}a_3b_2}\underline{e}_1 + \dots$$

$$\Longrightarrow \boxed{(a\wedge b)_i = \epsilon_{ijk}a_jb_k}$$

f) Combine a) and e).
i is a free suffix in e).

$$(\underline{b}\wedge\underline{c})_i = \epsilon_{ijk}b_jc_k$$

$$\boxed{a_i(\underline{b}\wedge\underline{c})_i = \underline{a}\cdot(\underline{b}\wedge\underline{c}) = \epsilon_{ijk}a_ib_jc_k}$$

W. Ex. 5.5 (A)

$$\boxed{(\underline{a} \wedge \underline{b})_i = \varepsilon_{ijk} a_j b_k} \Longrightarrow \{\nabla \wedge (\varphi \underline{v})\}_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varphi v_k)$$

$$= \varphi \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k + \varepsilon_{ijk} \left(\frac{\partial}{\partial x_j} \varphi\right) v_k$$

$$= \varphi (\nabla \wedge \underline{v})_i + \{(\nabla \varphi) \wedge \underline{v}\}_i$$

W. Ex. 5.5 $\boxed{\delta_{ij} a_j = a_i}$

(B) $\delta_{ip} \delta_{jq} c_p d_q$ p, q summed: $p \to i$, $q \to j$

$= c_i d_j$

(C) $\delta_{ij} b_{ij}$ both i,j summed: pick $i \to j$ or $j \to i$

$= b_{ii} = b_{jj}$

(D) $\delta_{ij} \varepsilon_{ijk}$ j summed $j \to i$

$= \varepsilon_{iik} = 0$.

W. Ex. 5.5 (E)

$$\{\nabla \wedge (\underline{\Omega} \wedge \underline{r})\}_i = \varepsilon_{ijk}\,\partial_j (\underline{\Omega} \wedge \underline{r})_k \qquad \underline{\Omega} = \text{const}$$

$$\underline{r} = (x_1, x_2, x_3)$$

$$= \varepsilon_{ijk}\,\partial_j\, \varepsilon_{klm} \Omega_l x_m$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm} \Omega_l\, \partial_j x_m \qquad \frac{\partial x_m}{\partial x_j} = \begin{cases} 1 & m = j \\ 0 & m \neq j \end{cases} = \delta_{jm}$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm}\, \Omega_l\, \delta_{jm} \quad m \to j$$

$$= \varepsilon_{ijk}\, \varepsilon_{klj}\, \Omega_l$$

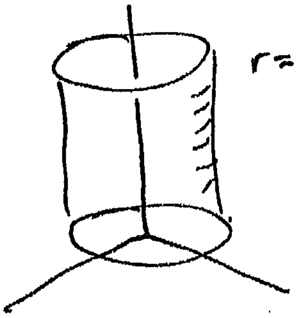
$$= (\delta_{il}\,\delta_{jj} - \delta_{ij}\,\delta_{jl})\,\Omega_l$$

$$= \delta_{il}\,\underbrace{\delta_{jj}}_{=3}\,\Omega_l - \underbrace{\delta_{ij}\,\delta_{jl}}_{l \to j \to i}\,\Omega_l \qquad l \to i$$

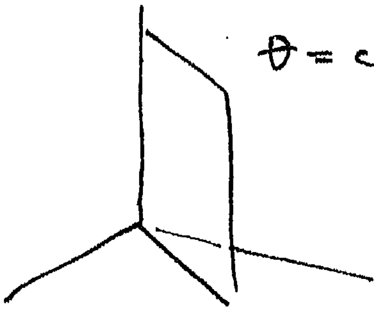
$$= 3\Omega_i - \Omega_i = 2\Omega_i$$

W. Ex 6.1 (A)

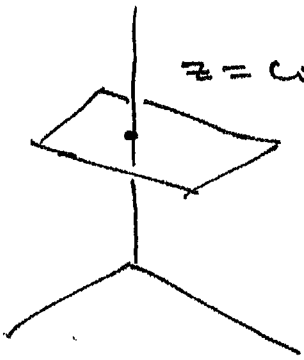
CPs



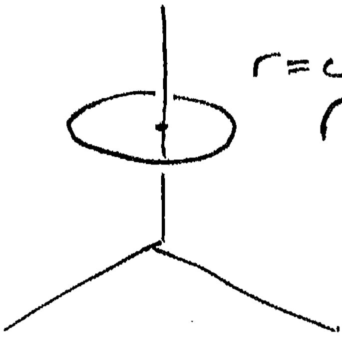
$r = \text{const}$



$\theta = \text{const}$

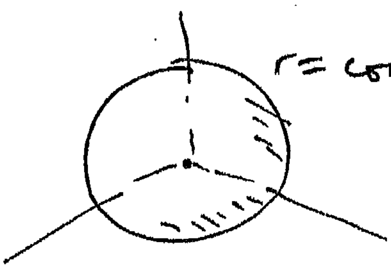


$z = \text{const}$

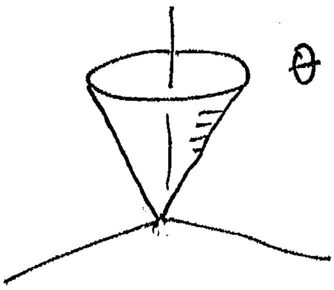


$r = \text{const} \cap z = \text{const}$.

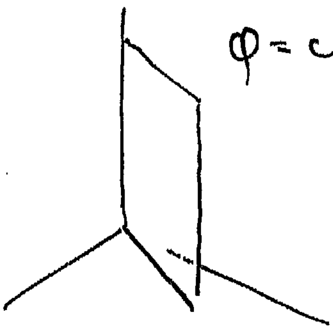
SPs



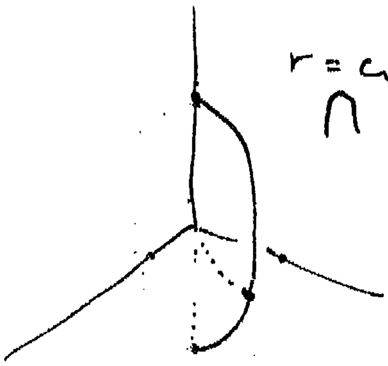
$r = \text{const}$



$\theta = \text{const}$

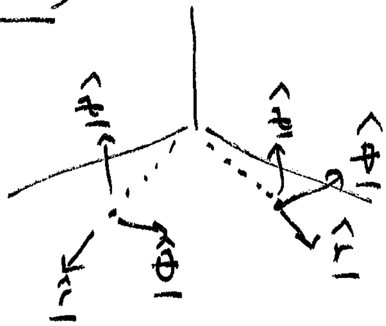


$\varphi = \text{const}$



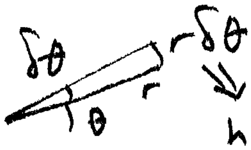
$r = \text{const} \cap \varphi = \text{const}$

W. Ex 6.1 (B)



W. Ex. 6.2

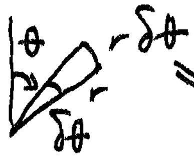
CPs (r, θ, z)



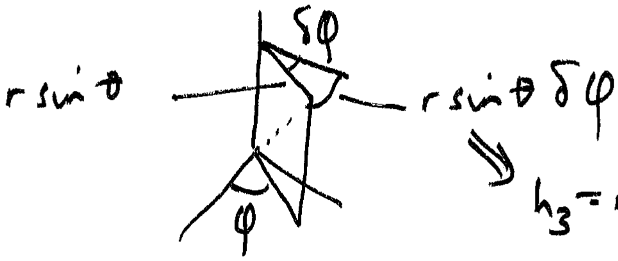
$\Rightarrow h_2 = r$

(r, z are already lengths)

SPs (r, θ, φ)



$\Rightarrow h_2 = r$



$\Rightarrow h_3 = r \sin\theta$

W. Ex 6.4

(A) Sphere, radius a, surface

$\longrightarrow$ SPs : $\alpha_1 = \text{const}$ $(r = a)$

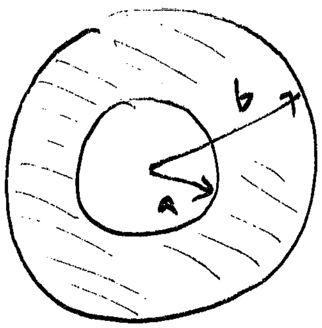
$h_2 = r, \quad h_3 = r \sin\theta$

$$\delta S = a^2 \sin\theta \, d\theta \, d\phi$$

$$A = \int dS = a^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= a^2 \cdot 2 \cdot 2\pi = 4\pi a^2.$$

(B)



$\rho(r) = \rho_0 a / r$

SPs: $h_1 = 1, \quad h_2 = r, \quad h_3 = r \sin\theta$

$$\delta V = r^2 \sin\theta \, \delta r \, \delta\theta \, \delta\phi$$

$$m = \int_V \rho \, dV$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=a}^{b} \frac{\rho_0 a}{r} \cdot r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2\pi \cdot \int_0^{\pi} \sin\theta \, d\theta \cdot \left[\rho_0 a \frac{r^2}{2} \right]_a^b$$

$$= 2\pi \cdot 2 \cdot \frac{\rho_0 a}{2} (b^2 - a^2)$$

$$= 2\pi \rho_0 a (b^2 - a^2)$$

W. Ex. 6.5 (A)

$\underline{F} = r\hat{\underline{r}}$, $S \equiv$ sphere, radius a.

Gauss' Thm., LHS

$$\int_S \underline{F}\cdot d\underline{S} = \int \underline{r}\cdot d\underline{S} = 4\pi a^3 \quad \text{(Example 6.4.2)}$$

Gauss' Thm., RHS

$$\nabla\cdot\underline{F} = \frac{1}{r^2}\partial_r\left(r^2 F_1\right)$$

Formulae handout $F_1 = r$, $F_2 = F_3 = 0$ (SPs)

$$= \frac{1}{r^2}\cdot 3r^2 = 3 .$$

[NOTE Cartesians: $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

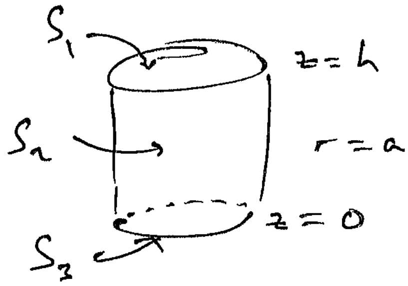
$\nabla\cdot\underline{r} = 1+1+1 = 3$ ✓

Now $$\int_V \nabla\cdot\underline{F}\, dV = 3\int dV = 3V = 3\cdot\frac{4}{3}\pi a^3$$

$$= 4\pi a^3$$

$\therefore$ LHS = RHS .

W. Ex 6.5 (B)



$\underline{F} = xz^3\hat{\underline{x}} + yz^3\hat{\underline{y}} + z^2(x^2+y^2)\hat{\underline{z}}$

Gauss' Thm, RHS

$$\nabla\cdot\underline{F} = z^3 + z^3 + 2z(x^2+y^2) = 2z^3 + 2zr^2$$

$$\int 2z(z^2+r^2)\,dV = \underline{\frac{\pi}{2}h^2a^2(h^2+a^2)} \quad \text{(EXAMPLE 6.4.1)}$$

LHS

$$\int_S \underline{F}\cdot d\underline{S} = \int_{S_1}\underline{F}\cdot d\underline{S}_1 + \int_{S_2}\underline{F}\cdot d\underline{S}_2 + \int_{S_3}\underline{F}\cdot d\underline{S}_3$$

• $\underline{\delta S}_1 = r\,\delta r\,\delta\theta\,\hat{\underline{z}}$ $\qquad \underline{F}\cdot\underline{\delta S}_1 = \underline{F}\cdot\hat{\underline{z}}\,r\,\delta r\,\delta\theta$

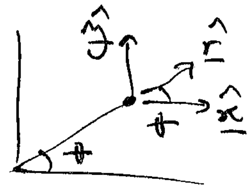
$$\underline{F}\cdot\hat{\underline{z}} = z^2(x^2+y^2) = h^2r^2 \leftarrow z=h \text{ on } S_1.$$

$$\int \underline{F}\cdot d\underline{S}_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} h^2r^2\cdot r\,dr\,d\theta$$

$$= 2\pi\cdot h^2\cdot\frac{1}{4}a^4 = \frac{\pi}{2}h^2a^4$$

• $\underline{\delta S}_2 = r\,\delta\theta\,\delta z\,\hat{\underline{r}}$

$$\underline{F}\cdot\hat{\underline{r}} = xz^3\,\underbrace{\hat{\underline{x}}\cdot\hat{\underline{r}}}_{\cos\theta} + yz^3\,\underbrace{\hat{\underline{y}}\cdot\hat{\underline{r}}}_{\sin\theta} + 0 \qquad \hat{\underline{z}}\cdot\hat{\underline{r}} = 0.$$



Also $x = r\cos\theta$, $y = r\sin\theta$

$$= z^3r\cos^2\theta + z^3r\sin^2\theta$$
$$= z^3r$$

$$\int \underline{F}\cdot d\underline{S}_2 = \int_{z=0}^{h}\int_{\theta=0}^{2\pi} z^3r\cdot r\,d\theta\,dz, \quad r=a \text{ on } S_2$$

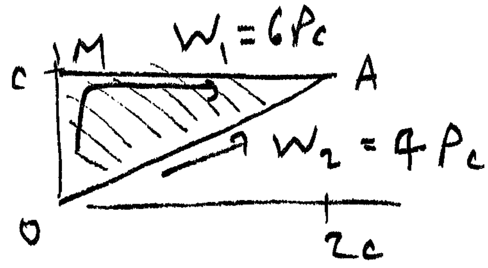
$$= 2\pi\cdot\frac{1}{4}h^4\cdot a^2 = \frac{\pi}{2}h^4a^2$$

• $\underline{\delta S}_3$: $z=0$ on $S_3 \Rightarrow \underline{F} = \underline{0} \Rightarrow \int \underline{F}\cdot d\underline{S}_3 = 0.$

Together $$\int \underline{F}\cdot d\underline{S} = \frac{\pi}{2}h^2a^4 + \frac{\pi}{2}h^4a^2 = \underline{\frac{\pi}{2}h^2a^2(a^2+h^2)}$$

W. Ex 6.6

Ex 3.2.2 :



$\underline{F} = \frac{P}{c}(3y, x, 0)$

Let $C = OAMO$, then

$$\oint_C \underline{F} \cdot \underline{dl} = W_2 - W_1 = -2Pc$$

Also, $\nabla \wedge \underline{F} = \frac{P}{c}\left(0, 0, \partial_x x - \partial_y(3y)\right) = -\frac{2P}{c}\underline{k}$

$$\underline{\delta S} = \underline{k}\,\delta S$$

$$\Rightarrow \int \nabla \wedge \underline{F} \cdot \underline{dS} = -\frac{2P}{c}\int dS$$

Area $= \frac{1}{2} \cdot 2c \cdot c$
$= c^2$

$$= -2Pc.$$

W. Ex. 8.2

$$\underline{u} = \frac{\kappa}{2\pi r}\,\hat{\underline{\theta}} \qquad (\text{CP}5)$$

C, circle, radius a

$$\Gamma = \oint_C \underline{u}\cdot d\underline{l} = \frac{\kappa}{2\pi a}\cdot 2\pi a = \kappa$$

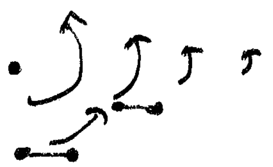
(any a.)

On S,

$$\underline{\omega} = \nabla\wedge\underline{u} = \underline{0} \qquad \left(\text{CP.}3 \quad F_2 = \frac{\kappa}{2\pi r}\right)$$

$$\Rightarrow \Gamma = \int_S \underline{\omega}\cdot d\underline{S} = 0 \quad ? \qquad (\neq \kappa \text{ above})$$

No : Singularity at $r=0$
Irrotational for $r>0$.
(Need to use Dirac δ-function).



W. Ex. 8.3

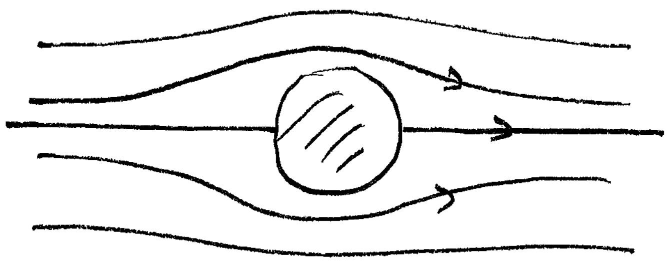
$$\psi = Uy - \frac{Ua^2y}{x^2+y^2}$$

CPs : $x^2+y^2=r^2$, $y = r\sin\theta$.

$\Rightarrow \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$.

For $r=a$, $\psi = 0$ (= const)

For $r \to \infty$, $\psi = Uy$ (= const for y = const)



cylinder, radius a.

For $\psi = Uy$, $\underline{u} = (\partial_y\psi, -\partial_x\psi, 0) = U\underline{i}$

W. Ex 9.2

Sps. $u_1 = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$

$u_2 = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$

$u_3 = 0$. (see SP.2)

$$\nabla\cdot\underline{u} = \frac{1}{r^2}\partial_r\left[U\left(r^2 - \frac{a^3}{r}\right)\cos\theta\right] - \frac{1}{r\sin\theta}\partial_\theta\left[U\left(1+\frac{a^3}{2r^3}\right)\sin^2\theta\right] + 0$$

$$= \frac{U}{r^2}\left(2r + \frac{a^3}{r^2}\right)\cos\theta - \frac{U}{r}\left(1+\frac{a^3}{2r^3}\right)\cdot 2\cdot\cos\theta$$

$$= 0.$$

Incompressible.

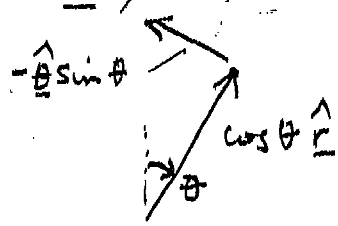
Boundary condition

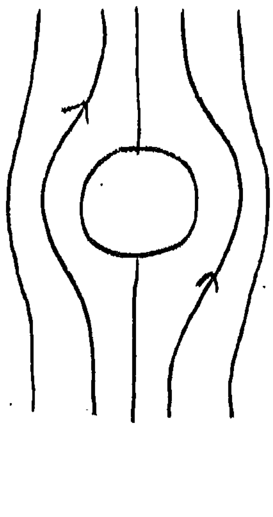
$\underline{u}\cdot\hat{\underline{n}} = \underline{u}\cdot\hat{\underline{r}} = u_1$

and $u_1 = 0$ on $r = a$

As $r \to \infty$, $\underline{u} \to U(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}})$

$= U\underline{k}$





W. Ex. 10.1

CPs, $\quad u_1 = u_3 = 0, \quad u_2 = \frac{k}{2\pi r}$

$$\underline{u} = \nabla \phi$$

$$\therefore \quad (\nabla \phi)_2 = \frac{1}{r} \partial_\theta \phi = \frac{k}{2\pi r}$$

$$\Rightarrow \phi = \frac{k}{2\pi} \theta + f(r, z) \, ,$$

but $u_1 = u_3 = 0 \Rightarrow$ take $f = 0$.

$$\therefore \quad \phi = \frac{k}{2\pi} \theta \, .$$

W. Ex. 10.3

$$\varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{k}{2\pi}\theta$$

Stagnation points occur when $\underline{u} = \underline{0}$, ie.

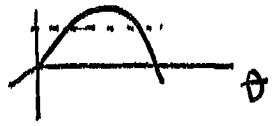
$$u_1 = U\cos\theta\left(1 - \frac{a^2}{r^2}\right) = 0 \quad (1)$$

&

$$u_2 = \frac{k}{2\pi r} - U\sin\theta\left(1 + \frac{a^2}{r^2}\right) = 0 \quad (2).$$

(1) $\Rightarrow$ $r = a$ or $\cos\theta = 0$

- $r = a$ & (2) $\Rightarrow$ $\sin\theta = \frac{k}{4\pi a U}$



2 solns for θ for $|k| < 4\pi a U$

1 $\quad =$

0 $\quad >$

- $\cos\theta = 0$. Take $\theta = -\frac{\pi}{2}$ $(k<0)$

(2) $\Rightarrow$ $\frac{k}{2\pi r} + U\left(1 + \frac{a^2}{r^2}\right) = 0$

$\times \frac{r^2}{U}$

$$r^2 + \frac{k}{2\pi U} r + a^2 = 0$$

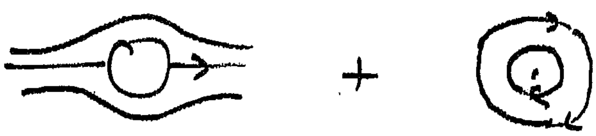
$$\left(r + \frac{k}{4\pi U}\right)^2 = \left(\frac{k}{4\pi U}\right)^2 - a^2$$

$$r = -\frac{k}{4\pi U} \pm \sqrt{\left(\frac{k}{4\pi U}\right)^2 - a^2}$$

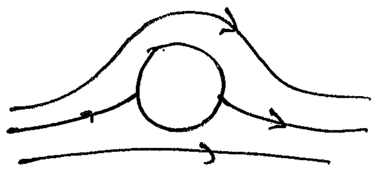
Have $\frac{|k|}{4\pi U} > a$ for a real soln.

Then $r > a$ for +ive root $(k<0)$.

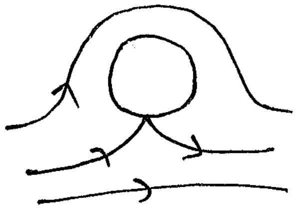
Sketch



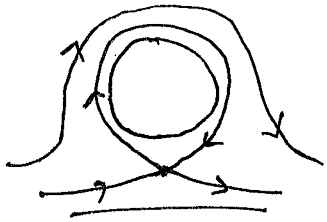
for $k<0$...



$|k| < 4\pi a U$



$|k| = 4\pi a U$



$|k| > 4\pi a U$

Bernoulli $\Rightarrow$ $p = -\frac{1}{2}\underline{u}\cdot\underline{u} + \text{const} = p_\infty + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho \underline{u}\cdot\underline{u}$

p is max at stagnation points $\longrightarrow$ net upward force.

On $r = a$,

$$p = p_\infty + \frac{1}{2}\rho U^2 - 2\rho U^2 \left(\frac{k}{4\pi a U} - \sin\theta\right)^2$$

Lift force $\underline{F}\cdot\underline{j} = -\int p\,\underline{j}\cdot d\underline{S} = -\iint p \sin\theta \, r \, d\theta \, dz$

Lift per unit length $F_l = -\int_0^{2\pi} p \sin\theta \cdot a \, d\theta$

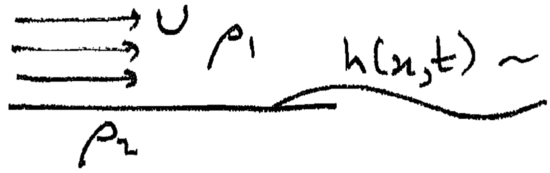
Note $\int_0^{2\pi} \sin\theta \, d\theta = 0$, $\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$, $\int_0^{2\pi} \sin^3\theta \, d\theta = 0$

$$\Longrightarrow F_l = -\int_0^{2\pi} 2\rho U^2 \cdot 2 \frac{k}{4\pi a U} \sin^2\theta \cdot a \, d\theta$$

$$= -\rho U k$$

Positive <u>lift</u> for <u>$k < 0$</u>.

W. Ex. 11.1



$h(x,t) \sim e^{ikx} e^{\sigma t}$

$$\underline{u}_1 = U\underline{i} + \nabla\varphi_1 \qquad |\nabla\varphi_j| << U, \quad j=1,2 .$$
$$\underline{u}_2 = \nabla\varphi_2$$

Let $\quad h(x,t) = \hat{h}\, e^{\sigma t + ikx}$
$\qquad \varphi_j(x,z,t) = \varphi_j(z)\, e^{\sigma t + ikx} \qquad$ (NORMAL MODES)

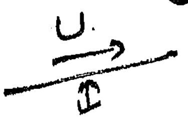
Now, $\quad \nabla\cdot\underline{u} = 0 \Rightarrow \nabla^2\varphi_j = 0$
$\qquad \Rightarrow (-k^2 + \partial_{zz})\varphi_j = 0 .$

Solutions $\quad \varphi_1 = A_1 e^{-kz}$, finite as $z \to \infty$ ①
$\qquad \varphi_2 = A_2 e^{kz}$, " $\quad z \to -\infty$

Fluid moves up and down with the interface

$$\hat{\underline{z}}\cdot\underline{u}_2 = \partial_z\varphi_2 = \partial_t h$$

... and along it



$$\hat{\underline{z}}\cdot\underline{u}_1 = \partial_z\varphi_1 = \partial_t h + U\partial_x h$$

Sub ① $\Rightarrow$

$$kA_2 = \sigma\hat{h} \qquad \Rightarrow \qquad \varphi_2 = \frac{\sigma}{k}\hat{h}e^{kz}$$
$$-kA_1 = (\sigma + Uki)\hat{h} \qquad \Rightarrow \qquad \varphi_1 = -\frac{1}{k}(\sigma + Uki)\hat{h}e^{-kz} \qquad ②$$

Pressure is continuous across the interface. Bernoulli $\Rightarrow$

$$\rho_1\left(f_1(t) - gh - \partial_t\varphi_1 - \tfrac{1}{2}|U\underline{i} + \nabla\varphi_1|^2\right)$$
$$= \rho_2\left(f_2(t) - gh - \partial_t\varphi_2 - \tfrac{1}{2}|\nabla\varphi_2|^2\right) \qquad ③$$

Drop $\frac{1}{2}|\nabla\varphi_2|^2$, as $|\nabla\varphi_j| << U$,

and $|U\underline{i}+\nabla\varphi|^2 = U^2 + 2U\partial_x\varphi_1 + \cancel{|\nabla\varphi|^2}$

For $\varphi_1 = \varphi_2 = h = 0$, the 'base flow' satisfies

$$\rho_1(f_1(t) - \tfrac{1}{2}U^2) = \rho_2 f_2(t). \quad (4)$$

(4) - (3) gives (LINEAR IN φ_j AND h)

$$\rho_1(gh + \partial_t\varphi_1 + U\partial_x\varphi_1) = \rho_2(gh + \partial_t\varphi_2)$$

Sub in (2) for φ_j and h,

$$\rho_1(g - \tfrac{1}{k}(\sigma + Uki)^2) = \rho_2(g + \tfrac{1}{k}\sigma^2)$$

$$(\rho_1+\rho_2)\sigma^2 + 2\rho_1 Uki\sigma = \rho_1 U^2k^2 + (\rho_1-\rho_2)gk \quad \text{(quadratic)}$$

$$\left(\sigma + \frac{\rho_1 Uki}{\rho_1+\rho_2}\right)^2 + \left(\frac{\rho_1 Uk}{\rho_1+\rho_2}\right)^2 = \frac{\rho_1}{\rho_1+\rho_2}U^2k^2 + \frac{\rho_1-\rho_2}{\rho_1+\rho_2}gk$$

$$\sigma = -\frac{\rho_1 Uk}{\rho_1+\rho_2}i \pm \left(\left(\frac{Uk}{\rho_1+\rho_2}\right)^2 \underbrace{(\rho_1(\rho_1+\rho_2) - \rho_1^2)}_{\rho_1\rho_2} + \frac{\rho_1-\rho_2}{\rho_1+\rho_2}gk\right)^{\frac{1}{2}}$$