Waves on a Stretched String

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With special thanx to .....!

Outline

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   - What is a “wave”?
   - Derivation of Governing PDE
   - D’Alembert’s solution and simple applications

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   - Standing waves
   - Principle of superposition
   - Some technical remarks

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Difficult to define precisely: here are two “definitions”.

Definition (Coulson, 1941:)

“We are all familiar with the idea of a wave; thus, when a pebble is dropped into a pond, water waves travel radially outwards; when a piano is played, the wires vibrate and sound waves spread throughout the room; when a radio station is transmitting, electric waves move through the ether. These are all examples of wave motion, and they have two important properties in common: firstly, **energy is propagated** to distant points; and secondly, the **disturbance travels** through the medium **without** giving the medium as a whole **any permanent displacement**.”

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Definition (Whitham, 1974:)

"...but to cover the whole range of wave phenomena it seems preferable to be guided by the intuitive view that **a wave is any recognizable signal** that is **transferred** from one part of the medium to another **with a recognizable velocity of propagation**."

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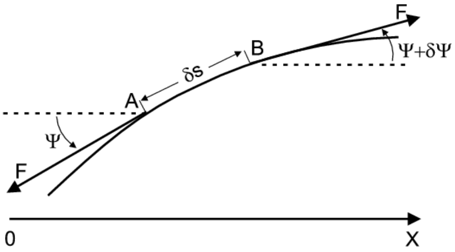
We begin with, perhaps, the simplest possible example.

A piece S of a string



- We suppose the string is under tension F, and that its mass per unit length is ρ. We consider transverse motion only ($\perp Ox$), and let the displacement be $y(x, t)$; we shall suppose y is small or -more precisely- we suppose $\mid \partial y/\partial x \mid \ll 1$ everywhere.

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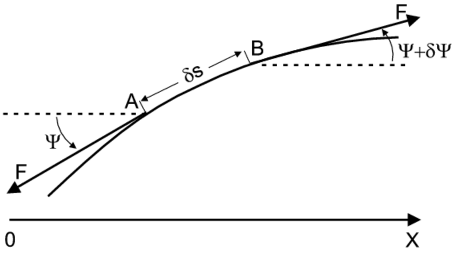

- Longitudinal motion negligible $\Rightarrow$ F is independent of x (see part *ii* below). We also take ρ independent of x.

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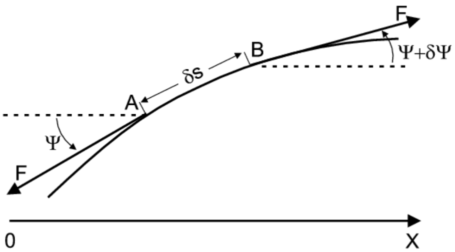
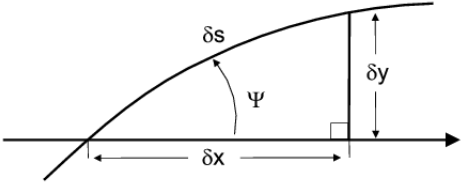


- Apply N2 to a small element of the string AB of length δs.

$$\rho \delta s \frac{\partial^2 y}{\partial t^2} = F\{sin(\psi + \delta\psi) - sin\psi\}. \qquad (1)$$

Local geometry of string S



Now, from sketch Fig. 7

$$\delta s^2 \approx \delta x^2 + \delta y^2 \Rightarrow \delta s \approx \left\{ 1 + \left(\frac{\partial y}{\partial x} \right)^2 \right\}^{1/2} \delta x \qquad (2)$$

Local geometry of string S

Therefore, because $\mid \partial y/\partial x \mid \ll 1 \; \forall \; x$ (by assumption),

$$\delta s \approx \delta x \tag{3}$$

to highest order. Likewise

$$\tan\psi = \partial y/\partial x \ll 1 \Rightarrow \psi \approx \partial y/\partial x,$$

and, in Eq. (1),

$$\begin{aligned}\sin(\psi+\delta\psi)-\sin\psi &\approx \cos\psi\cdot\delta\psi \\ &\approx \{1+\tan^2\psi\}^{-1/2}\delta\psi \\ &\approx \delta\psi \\ &\approx \delta(\partial y/\partial x) \\ &\approx (\partial^2 y/\partial x^2)\delta x.\end{aligned}$$

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Thus Eq. (1) becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{\rho}\frac{1}{\delta x}\frac{\partial^2 y}{\partial x^2}\delta x = \frac{F}{\rho}\frac{\partial^2 y}{\partial x^2}. \quad (4)$$

Finally we have

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad (5)$$

where the constant c satisfies

$$c^2 = \frac{F}{\rho}. \quad (6)$$

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• (*ii*) We have assumed F is uniform. Hooke’s Law $\Rightarrow$ change in $F \propto$ change in length. But

$$\begin{aligned} \text{change in length} &= \delta s - \delta x \\ &\approx \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{1/2} \delta x - \delta x \\ &\approx \left\{1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right\} \delta x \\ &= \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \delta x \end{aligned}$$

which is second-order in small quantities $\Rightarrow$ the assumption of uniform F is OK.

Some comments... Kinetic Energy

- (*iii*) The kinetic energy (KE) of an element of length δs is

$$\frac{1}{2}\rho\delta s\left(\frac{\partial y}{\partial t}\right)^2 \approx \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2\delta x,$$

which implies that the KE between $x = a$ and $x = b\ (> a)$ is

$$\mathrm{KE} = T = \frac{1}{2}\rho\int_a^b\left(\frac{\partial y}{\partial t}\right)^2 dx. \qquad (7)$$

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The potential energy (PE) of an element of length δs is

$$\begin{aligned} F \times \text{increase in length} &= F(\delta s - \delta x) \\ &\approx \frac{1}{2} F \left(\frac{\partial y}{\partial x} \right)^2 \delta x \quad (\text{from}(ii)). \end{aligned}$$

Thus the PE between $x = a$ and $x = b$ ($> a$) is

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NB

T, V are second-order in small quantities, i.e. $(\partial y/\partial x)^2$, $(\partial y/\partial t)^2$, whereas the wave equation Eq. (5) itself is first-order.

D’Alembert’s general solution

• Unusually we can find the general solution of the wave equation Eq. (5). Change variables from (x, t) to (u, v), where

$$u = x - ct, \qquad v = x + ct. \tag{9}$$

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Chain rule $\Rightarrow$

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$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} = y_u + y_v \Rightarrow$$

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(y_u + y_v) = y_{uu} + 2y_{uv} + y_{vv},$$

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- Substitute in the wave equation Eq. (5)

$$c^2(y_{uu} + 2y_{uv} + y_{vv}) = c^2(y_{uu} - 2y_{uv} + y_{vv})$$

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Therefore,

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = 0 \Rightarrow \frac{\partial y}{\partial v} = g_\star(v),$$

where $g_\star$ is any function[1] $\Rightarrow$

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D’Alembert’s general solution

$$y = \underbrace{\int^{v} g_{\star}(s)ds}_{g(v)} + f(u),$$

where f is any function[1]. Thus

$$y = f(u) + g(v),$$

i.e.

$$y = f(x - ct) + g(x + ct). \tag{11}$$

Eq. (11) is d’Alembert’s solution (the general solution) of the wave equation (5), first published in 1747 [J. le Rond d’Alembert (1717-83)].

D’Alembert’s general solution

• The functions f and g in Eq. (11) are determined by the boundary and initial conditions.

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To begin with, suppose that, at $t = 0$,

$$y(x,0) = \phi(x), \quad \dot{y}(x,0) = 0. \tag{12}$$

Thus the string is initially at rest $\forall\ x$, but has a displacement given by $y = \Phi(x)$.

D’Alembert’s general solution

From (11) and (12) we must have

$$f(x) + g(x) = \Phi(x), \quad -cf'(x) + cg'(x) = 0.$$

where $'$ denotes “derived function”. The second (RHS) gives $f'(x) = g'(x) \Rightarrow f(x) = g(x) + \alpha$, where α is a constant. The first (LHS) then gives:

$$f(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\alpha, \quad g(x) = \frac{1}{2}\Phi(x) - \frac{1}{2}\alpha.$$

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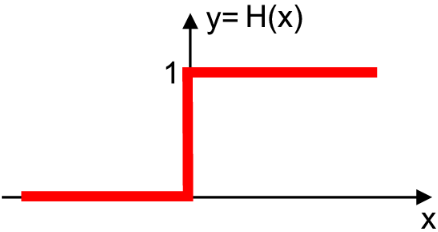


Example 1

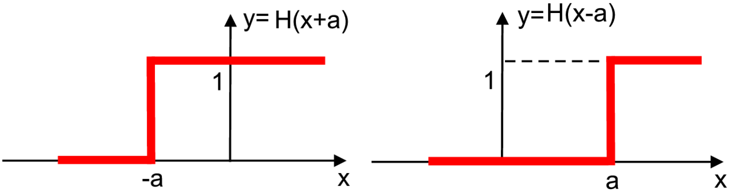
At $t = 0$, an infinite string is at rest and

$$y(x, 0) = b\{H(x + a) - H(x - a)\}, \tag{15}$$

where $a, b > 0$ constants. Find $y(x, t)$ for $\forall\ x, t$ and sketch your solution.

Example 1: Solution

Shifted Heaviside functions



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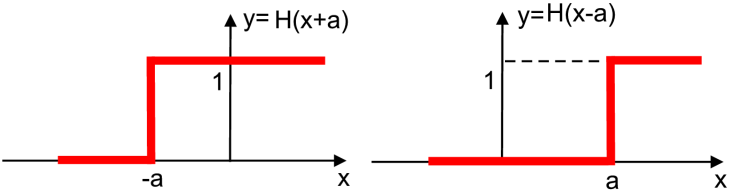

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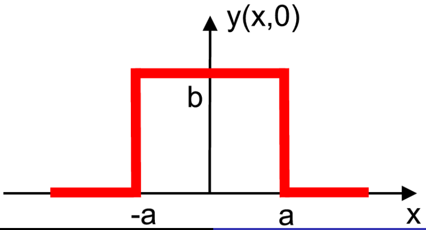

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$$y(x,t) = \frac{b}{2}\{H(x-ct+a)-H(x-ct-a)\} + \frac{b}{2}\{H(x+ct+a)-H(x+ct-a)\} \quad (16)$$

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- (*i*) its height is $(1/2)b$, not b, and
- (*ii*) its end points are $(ct-a, ct+a)$, not $(-a, a)$.

This is a signal with graph like Fig. 1 except for (*i*) and (*ii*).

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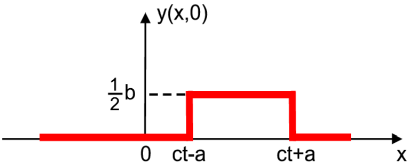
Thus the first term in Eq. (16) has graph of travelling signal to right with speed c:



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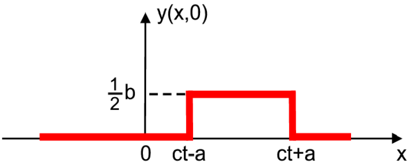
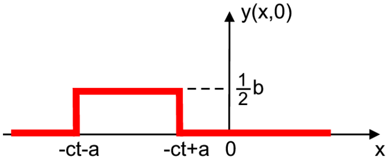


Likewise the second term has graph of travelling sigal to left with speed c:



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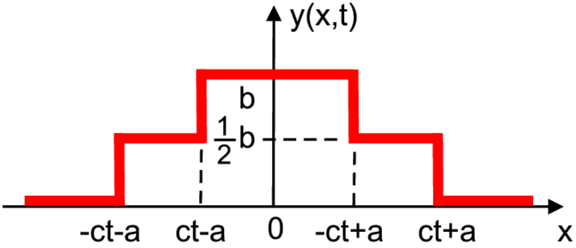

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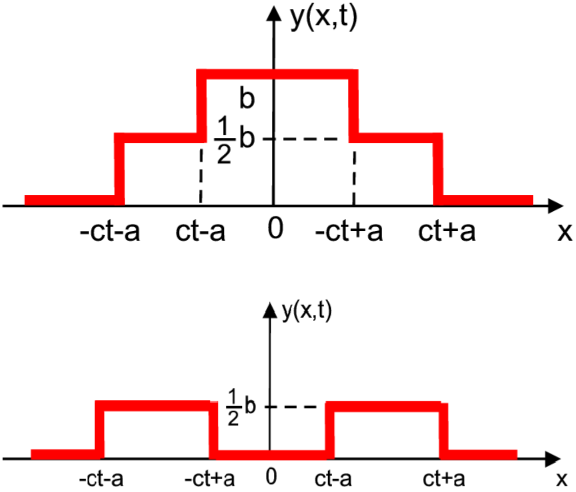

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Likewise $g(x + ct)$ is a profile of constant shape and size that moves to the left with speed c. Each is a travelling wave (or progressive wave).
In the above example, the initial profile splits into two; one half travels to the right, one half to the left.

Example 2

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Solution

From Eq. (13) $\Rightarrow$

$$y(x,t) = \frac{1}{2}a\left\{\sin[k(x-ct)] + \sin[k(x+ct)]\right\}. \qquad (17)$$

We shall revisit Eq. (17) soon.

Initially moving string

• More general than Eq. (12) is the case when the string is also moving at $t = 0$.

$$y(x,0) = \Phi(x), \qquad y_t(x,0) = \Psi(x). \tag{18}$$

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$$f(x) + g(x) = \Phi(x), \qquad -cf'(x) + cg'(x) = \Psi(x).$$

The second condition (RHS) gives

$$f'(x) - g'(x) = (-1/c)\Psi(x)$$

$\Rightarrow$

Initially moving string: General solution

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Thus

$$\begin{aligned} f(x) &= \frac{1}{2}\Phi(x) - \frac{1}{2c}\int_d^x \Psi(s)ds, \\ g(x) &= \frac{1}{2}\Phi(x) + \frac{1}{2c}\int_d^x \Psi(s)ds, \end{aligned}$$

and from Eq. (11) $\Rightarrow$

Initially moving string: General solution

$$
\begin{aligned}
y(x,t) &= \frac{1}{2}\{\Phi(x-ct)+\Phi(x+ct)\} \\
&+ \frac{1}{2c}\int_d^{x+ct}\Psi(s)ds - \frac{1}{2c}\int_d^{x-ct}\Psi(s)ds
\end{aligned}
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&+ \frac{1}{2c}\int_d^{x+ct}\Psi(s)ds - \frac{1}{2c}\int_d^{x-ct}\Psi(s)ds
\end{aligned}
$$

$\Rightarrow$

$$
y(x,t) = \frac{1}{2}\{\Phi(x-ct)+\Phi(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct}\Psi(s)ds. \quad (19)
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Example 3

Given that $\Phi(x) = a\cos(kx)$, $\Psi(x) = -kca\sin(kx)$ in Eq. (18), find $y(x, t)$.

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Solution

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$$
\begin{aligned}
y(x,t) &= \frac{a}{2}\{\cos(k(x-ct)) + \cos(k(x+ct))\} - \frac{ka}{2}\int_{x-ct}^{x+ct} \sin(ks)ds \\
&= \frac{a}{2}\{\cos(k(x-ct)) + \cos(k(x+ct))\} + \frac{a}{2}[\cos(ks)]_{x-ct}^{x+ct} \\
&= a\cos(k(x+ct))
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Thus the two terms in Eq. (19) combine so that the wave is purely travelling to the left.

Exercises for students:

- Show that Eq. (19) gives a wave travelling only to the left (i.e. $y = g(x + ct)$) if and only if $\Psi(x) = c\Phi'(x)$.

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- What initial conditions give

$$y(x,t) = a\tanh(k(x - ct))$$

for

$$-\infty < x < \infty \text{ and } \forall t \geq 0?$$

Outline

1. Waves on Infinite Strings
   - What is a “wave”?
   - Derivation of Governing PDE
   - D’Alembert’s solution and simple applications
2. Strings of Finite Length
   - Standing waves
   - Principle of superposition
   - Some technical remarks

Standing waves

- Now Eq. (17)

$$y(x,t) = \frac{1}{2} a \left\{ \sin[k(x - ct)] + \sin[k(x + ct)] \right\}.$$

can be written

$$(\text{since} \quad \sin A + \sin B = 2 \sin\left[\frac{A+B}{2}\right] \cos\left[\frac{A-B}{2}\right])$$

$$y(x,t) = a \sin(kx) \cos(kct) \tag{20}$$

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What is Eq. (20) describing?

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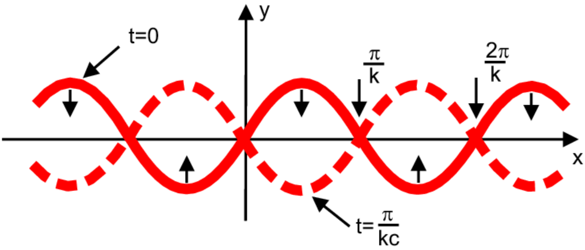
What is Eq. (20) describing? Let's take a snapshot...!

Standing waves



Thus y is always zero at $x = n\pi/k$.
Between $x = r_1\pi/k$ and $x = r_2\pi/k$ the string oscillates periodically in time.
Eq. (20) is an example of a standing wave, with a being the amplitude, k the wavenumber ($k > 0$), $2\pi/k$ the wavelength.
The period of oscillation is $2\pi/kc$.

Method of separation of variables

• Standing waves occur with a string of finite length L. Suppose the string is fixed at $x = 0$, $x = L$ (e.g., a piano wire or violin) so the solution of Eq. (5), the wave equation, must satisfy

$$y(0, t) = y(L, t) = 0. \tag{21}$$

We look for solutions of Eq. (5) of the form (separable solutions)

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Substituting in Eq. (5) $\Rightarrow$

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$$c^2 X'' T = X\ddot{T}$$

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Method of separation of variables

$$\frac{X''}{X} = \frac{1}{c^2}\left(\frac{\ddot{T}}{T}\right).$$

The LHS depends only on x, the RHS depends only on t so the equation can be true for $\forall\ (x, t)$ only if .

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There are three cases to consider.

Case 1

[1] Constant $> 0 = k^2$

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From Eq. (21):

$$y(0,t) = 0 \quad \Rightarrow \quad A = 0 \Rightarrow y = B\sin(kx)$$

$$y(L,t) = 0 \quad \Rightarrow \quad B\sin(kL) = 0.$$

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and

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$\Rightarrow$

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For useful/interesting results we cannot have $B = 0$ which implies $sin(kl) = 0 \Rightarrow kL = n\pi$ (n$=0, 1, 2...$)

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$$X = B_n sin(n\pi x/L)$$

and

$$\ddot{T} = -(n\pi c/L)^2 T.$$

$\Rightarrow$

$$T = \alpha\cos(n\pi ct/L) + \beta\sin(n\pi ct/L).$$

Summary of a general solution

Thus a solution of Eq. (5) (wave equation) of the form Eq. (22) (separable solutions) satisfying Eq. (21) (fixed boundary) is

$$y = \sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \quad (n = 1, 2, 3...). \tag{24}$$

For each n, the solution in Eq. (24) is a periodic wave [like Eq. (20)] with period $2\pi L/n\pi c = 2L/nc$.

Nomenclatura

We often rewrite

$\cos(n\pi ct/L)$ as $\cos(\omega_n t)$

$\sin(n\pi ct/L)$ as $\sin(\omega_n t)$

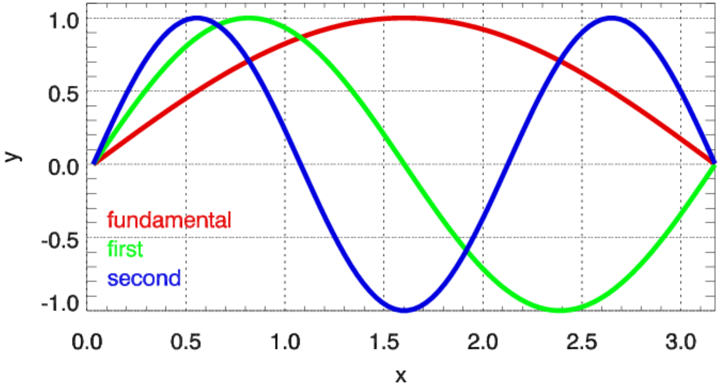
where ω_n is the angular frequency:

$$\omega_n = \frac{n\pi c}{L}. \tag{25}$$

Definition (Normal mode)

Each of the solutions in Eq. (24) is a normal mode of vibration.

Normal modes



Standing fundamental, 1st, and 2nd harmonics.

Full general solution: Superposion

• Since Eq. (5) is a linear equation so any linear combination of the solutions in Eq. (24) is also a solution. This is the principle of superposition. Thus

$$y = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \quad (26)$$

is a solution of Eq. (5) satisfying Eq. (21). It is in fact the general solution of Eq. (5)-(21); the constants α_n, β_n are determined by the initial conditions (see next Chapter).

Question: In general, is this solution periodic in time? Explain your answer.

Complex notation

- Consider the real part, $\Re$, of the complex quantity

$$A\exp[i(kx-\omega t)],$$

where k and ω are real but

$$A = A_r + iA_i$$

is complex. Now

$$\begin{aligned} \Re\{A\exp[i(kx-\omega t)]\} &= A_r\cos(kx-\omega t) - A_i\sin(kx-\omega t) \\ &= \sqrt{A_r^2+A_i^2}\cos[(kx-\omega t)+\epsilon] \end{aligned}$$

where

$$\cos\epsilon = A_r/\sqrt{A_r^2+A_i^2}, \qquad \sin\epsilon = A_i/\sqrt{A_r^2+A_i^2}.$$

Complex notation

We shall consider situations in which the dependent variable, say ϕ, has the form

$$\phi = \alpha \cos[(kx - \omega t) + \epsilon]$$

(or with sin instead of cos).

Note: $\phi = \sin kx[(-\alpha \sin \epsilon) \cos \omega t + (\alpha \cos \epsilon) \sin \omega t] + \cos kx[(-\alpha \cos \epsilon) \cos \omega t + (\alpha \sin \epsilon) \sin \omega t]$,
and the first term is equivalent to Eq. (24).

In linear problems it is often convenient to write (A complex; k, ω real)

$$\phi = A \exp[i(kx - \omega t)]; \tag{27}$$

we do of course really mean the real part of Eq. (27) but many problems can be solved most easily by working directly with Eq. (27) and **only taking the real part** right at the end.

Complex notation

In Eq. (27), k is again the wavenumber and ω is the angular frequency.

To satisfy the 1D wave equation Eq. (5), $\omega = kc$. The period is $2\pi/\omega$ and the frequency is $\omega/2\pi$. The frequency, measured in s^{-1} (Hz, hertz), is the number of complete oscillations that the wave makes during 1 sec at a fixed position. Finally,

$$|A| = \sqrt{A_r^2 + A_i^2}$$

is the amplitude. Eq. (27) is a periodic or harmonic wave.