Use of Fourier Series

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With special thanx to .....!

Outline

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Aims

- In § 1.4, we saw that a solution of the PDE for $y(x,t)$

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad \text{with} \qquad y(0,t) = y(L,t) = 0, \qquad (1)$$

is Eq (1.24), viz.

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right]. \qquad (2)$$

Eq. (2) is in fact the general solution of (1).

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Eq. (2) is in fact the general solution of (1).

Our aim is to show how the constants $\{\alpha_n\}$ and $\{\beta_n\}$ can be determined, and to indicate some extensions.

Importance of ICs

• The constants $\{\alpha_n\}$ and $\{\beta_n\}$ are determined uniquely by the initial conditions, i.e. the value of $y(x,0)$ and $\dot{y}(x,0)$ (or, more generally, by the values of $y(x,t_0)$, $\dot{y}(x,t_0)$ for any t_0). In any particular motion, the values of $y(x,0)$ and $\dot{y}(x,0)$ can be chosen independently of one another.

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- We note from Eq. (2) that

$$\begin{aligned}\dot{y}(x,t) &= \sum_{n=1}^{\infty}\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\\ &\times \left[-\alpha_n \sin\left(\frac{n\pi ct}{L}\right)+\beta_n\cos\left(\frac{n\pi ct}{L}\right)\right]. \qquad (3)\end{aligned}$$

ICs

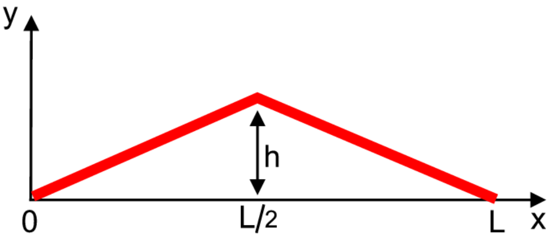
- Thus, from Eqs. (2) and (3)

$$y(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right), \tag{4a}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \beta_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{4b}$$

Example 1: Plucked string with length L

Find $\{\alpha_n\}$, $\{\beta_n\}$ for the case of a plucked string of length L, with its ends fixed, released from rest when the midpoint is drawn aside through a distance h. Thus



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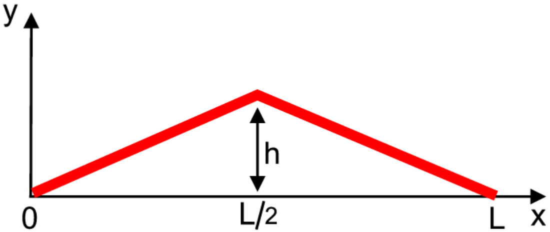

$$y(x, 0) = \qquad (6)$$

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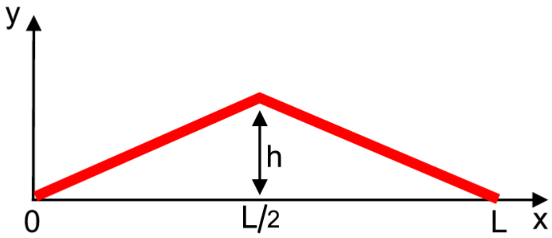

$$y(x,0) = \begin{cases} \frac{2h}{L}x & (0 \leq x \leq \frac{1}{2}L) \\ \frac{2h}{L}(L-x) & (\frac{1}{2}L \leq x \leq L) \end{cases} \tag{5}$$

and

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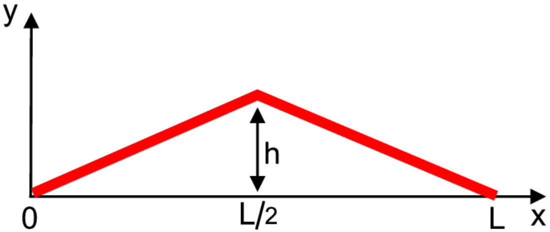

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Solution

- Comparing Eqs. (6) and (4b), we can reconcile them by taking

$$\beta_n = \quad . \tag{7}$$

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and integrate from $x = 0$ to $x = L$. Thus

$$\int_0^L y(x,0)\sin\left(\frac{m\pi x}{L}\right)dx =$$

$$\sum_{n=1}^{\infty}\alpha_n\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{8}$$

Technical lemma

- Consider I_{mn}, where

$$I_{mn} = \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{9}$$

Question:

How to proceed with the evaluation of I_{mn}?

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How to proceed with the evaluation of I_{mn}?

Now we need to linearise the integrand

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

for $\forall$ A and B, so the integrand in Eq. (9) is

$$\frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right].$$

Since $m, n > 0$ are integers, there are two possible cases.

Technical lemma: Evaluate I_{nm}

$m \neq n$:

$$I_{mn} = \frac{l}{2\pi}\left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{(m-n)} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{(m+n)}\right]_0^L =$$

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$m = n$:

$$I_{mn} = \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2m\pi x}{L}\right)\right] dx = \frac{L}{2}.$$

$\Rightarrow$

$$I_{mn} = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n. \end{cases} \tag{10}$$

Formula for $\{\alpha_m\}$

• Hence the RHS of Eq. (8) reduced to $\frac{L}{2}\alpha_m$, and Eq. (8) can be then rewritten

$$\alpha_m = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{m\pi x}{L}\right)dx, \tag{11}$$

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In our particular case, use of Eq. (5) gives

Formula for $\{\alpha_m\}$ for Example 1

$$
\begin{aligned}
\alpha_m &= \frac{4h}{L^2}\int_0^{L/2} x\sin\left(\frac{m\pi x}{L}\right)dx + \frac{4h}{L^2}\int_{L/2}^{L}(L-x)\sin\left(\frac{m\pi x}{L}\right)dx,\\
&= \frac{4h}{Lm\pi}\left\{\left[-x\cos\left(\frac{m\pi x}{L}\right)\right]_0^{L/2} + \int_0^{L/2}\cos\left(\frac{m\pi x}{L}\right)dx\right\} +\\
&\quad \frac{4h}{Lm\pi}\left\{\left[-(L-x)\cos\left(\frac{m\pi x}{L}\right)\right]_{L/2}^{L} - \int_{L/2}^{L}\cos\left(\frac{m\pi x}{L}\right)dx\right\},\\
&= -\frac{2h}{m\pi}\cos\left(\frac{m\pi}{2}\right) + \frac{2h}{m\pi}\cos\left(\frac{m\pi}{2}\right) + \frac{4h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)\\
&\quad + \frac{4h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right) = \frac{8h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right).
\end{aligned}
$$

Formula for $\{\alpha_m\}$ and solution for Example 1

Thus

$$\alpha_m = \begin{cases} 0 & (m = 2p) \\ \frac{8h(-1)^p}{\pi^2(2p+1)^2} & (m = 2p+1). \end{cases} \qquad (12)$$

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• Use of Eqs. (7) (β_m) and (12) (α_m) upon substitution into Eq. (2) gives the

Time-dependent full solution of a plucked oscillating finite string:

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Time-dependent full solution of a plucked oscillating finite string:

$$y(x,t) = \frac{8h}{\pi^2}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)^2}\sin\left(\frac{(2p+1)\pi x}{L}\right) \times \cos\left(\frac{(2p+1)\pi ct}{L}\right). \tag{13}$$

Example 2: Initially moving string

Quest

Find $\{\alpha_n\}$, $\{\beta_n\}$ for the case of a string of length L, given that Eq. (1) holds and in addition

$$y(x,0) = 0$$

and

$$\dot{y}(x,0) = 4Vx(L-x)/L^2.$$

Solution

- Determine α_n:

In this case $y(x,0) = 0 \Rightarrow$

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- Determine β_n:

Then from Eq. (4b) $\Rightarrow$

$$\int_0^L \frac{4Vx(L-x)}{L^2} \sin\left(\frac{m\pi x}{L}\right) dx =$$

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Then from Eq. (4b) $\Rightarrow$

$$\int_0^L \frac{4Vx(L-x)}{L^2} \sin\left(\frac{m\pi x}{L}\right) dx = \beta_m \left(\frac{m\pi c}{L}\right) \frac{L}{2},$$

using the same technique that leads from Eq. (8) to Eq. (11).

Solution: Determining $\{\alpha_n\}$ and $\{\beta_n\}$

Hence

$$\beta_m \frac{m\pi c}{2} = \frac{4V}{L^2} \left\{ \underbrace{\left[-\frac{L}{m\pi} x(L-x) \cos\left(\frac{m\pi x}{L}\right) \right]_0^L}_{=0} + \frac{L}{m\pi} \int_0^L (L-2x) \cos\left(\frac{m\pi x}{L}\right) dx \right\},$$

$$= \frac{4V}{L^2} \left\{ 0 + \frac{L}{m\pi} \int_0^L (L-2x) \cos\left(\frac{m\pi x}{L}\right) dx \right\}.$$

$\Rightarrow$

Solution: Determining $\{\alpha_n\}$ and $\{\beta_n\}$

$$\beta_m = \frac{8V}{m^2\pi^2 cL}\left\{\left[\frac{L}{m\pi}(L-2x)\sin\left(\frac{m\pi x}{L}\right)\right]_0^L + \frac{2L}{m\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right) dx\right\},$$

$$=$$

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$$\beta_m = \frac{8V}{m^2\pi^2 cL}\left\{\left[\frac{L}{m\pi}(L-2x)\sin\left(\frac{m\pi x}{L}\right)\right]_0^L + \frac{2L}{m\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right)dx\right\},$$

$$= \frac{16VL}{m^4\pi^4 c}\left[\cos\left(\frac{m\pi x}{L}\right)\right]_L^0$$

$$=$$

Solution: Determining $\{\alpha_n\}$ and $\{\beta_n\}$

$$
\begin{aligned}
\beta_m &= \frac{8V}{m^2\pi^2 cL}\left\{\left[\frac{L}{m\pi}(L-2x)\sin\left(\frac{m\pi x}{L}\right)\right]_0^L \right.\\
&\qquad \left. +\frac{2L}{m\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right)dx\right\},\\
&= \frac{16VL}{m^4\pi^4 c}\left[\cos\left(\frac{m\pi x}{L}\right)\right]_L^0\\
&= \frac{16VL}{m^4\pi^4 c}[1-(-1)^m].
\end{aligned}
$$

Solution: Summary for Example 2

Thus

$$\beta_m = \begin{cases} 0 & (m = 2p) \\ \frac{32VL}{\pi^4 c(2p+1)^4} & (m = 2p+1) \end{cases}, \qquad (15)$$

and so

$$y(x,t) = \frac{32VL}{\pi^4 c} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} \sin\left(\frac{(2p+1)\pi x}{L}\right) \times \sin\left(\frac{(2p+1)\pi ct}{L}\right). \qquad (16)$$

Quest

Compare the rates of fall-off with p of the coefficients in Eqs. (13) and (16).

Reformulate the general solution

- Consider a string occupying $0 \leq x \leq L$ with $y(0,t) = 0$, $y(L,t) = 0$, and consider the normal mode Eq. (1.24)

$$y = \sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\}.$$

We rewrite this in the form (with $A_n \geq 0$, $0 \leq \epsilon_n \leq 2\pi$):

$$y = A_n \cos\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \sin\left(\frac{n\pi x}{L}\right). \tag{17}$$

In Eq. (17) A_n is the amplitude and ϵ_n is the phase.

Reformulate the general solution

Note:

$$A_n \cos\{n\pi ct/L + \epsilon_n\} = A_n \cos\epsilon_n \cos(n\pi ct/L) - A_n \sin\epsilon_n \sin(n\pi ct/L)$$

so Eqs. (1.24) and (17) are the same **provided**

$$A_n \cos\epsilon_n = \alpha_n, \qquad A_n \sin\epsilon_n = -\beta_n.$$

$\Rightarrow$

$$A_n^2(\cos^2\epsilon_n + \sin^2\epsilon_n) = \alpha_n^2 + \beta_n^2$$

$\Rightarrow$

$$A_n = +\sqrt{\alpha_n^2 + \beta_n^2}, \qquad \tan\epsilon_n = -\beta_n/\alpha_n.$$

Kinetic energy T_n

- By Eq. (1.7), the kinetic energy T_n associated with Eq. (17) is

$$T_n = \frac{1}{2}\rho A_n^2 \left(\frac{n\pi c}{L}\right)^2 \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx,$$

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$\Rightarrow$

$$T_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \qquad (18)$$

Potential energy V_n

Likewise, by Eq. (1.8), the potential energy V_n associated with Eq. (17) is

$$\begin{aligned} V_n &= \frac{1}{2} F A_n^2 \left(\frac{n\pi}{L}\right)^2 \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx, \\ &= \frac{F\pi^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \end{aligned}$$

From Eq. (1.6) $F = \rho c^2$, $\Rightarrow$

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&= \frac{F\pi^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}.
\end{aligned}
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From Eq. (1.6) $F = \rho c^2$, $\Rightarrow$

$$
V_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \qquad (19)
$$

Total modal energy E_n

The total modal energy therefore $E_n = T_n + V_n$ is given by

$$E_n = \frac{\rho \pi^2 c^2 n^2 A_n^2}{4L} = \frac{\rho L}{4} \omega_n^2 A_n^2, \qquad \omega_n = \frac{n\pi c}{L}, \tag{20}$$

where ω_n is the angular frequency of this normal mode.

$\Rightarrow E_n \propto A_n^2$ and $E_n \propto \omega_n^2$.

Note: $E_n \propto A_n^2$ indicates that a much bigger proportion of the total energy is contained in the first few modes of, say, Eq. (16) than in the same number of modes of, say, Eq. (13).

Check it!

Total energy E

- Now consider the general motion given by Eq. (1.24).

Recall the lemmae!

Because, for $m \neq n$,

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \quad =$$

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \quad = \qquad ,$$

Total energy E

- Now consider the general motion given by Eq. (1.24).

Recall the lemmae!

Because, for $m \neq n$,

$$\begin{aligned} \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx &= 0 \\ \int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx &= 0, \end{aligned}$$

(the first leads to Eq. (10b) and the second is proved likewise), it follows immediately that

Total energy E

$$T = \sum_{n=1}^{\infty} T_n = \frac{\rho\pi^2 c^2}{4L} \sum_{n=1}^{\infty} n^2 A_n^2 \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\},$$

$$V = \sum_{n=1}^{\infty} V_n = \frac{\rho\pi^2 c^2}{4L} \sum_{n=1}^{\infty} n^2 A_n^2 \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\},$$

$$E = T + V = \frac{\rho\pi^2 c^2}{4L} \sum_{n=1}^{\infty} n^2 A_n^2. \qquad (21)$$

Total energy E of the plucked string

- Apply to **Example 1** (plucked string). From Eq. (12) we have

$$\begin{aligned} A_{2n} &= 0 \\ A_{2n+1} &= \frac{8h}{\pi^2(2n+1)^2}. \end{aligned}$$

Thus, from the last of Eq. (21),

$$E = \qquad (22)$$

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Thus, from the last of Eq. (21),

$$E = \frac{\rho\pi^2 c^2}{4L}\sum_{n=0}^{\infty}\frac{64(2n+1)^2 h^2}{\pi^4(2n+1)^4},$$

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and thus $\Rightarrow$

$$E = \frac{16\rho h^2 c^2}{\pi^2 L} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}. \qquad (22)$$

Total energy E of the plucked string

Amazingly it turns out that we can evaluate the infinite series in Eq. (22) by using Eq. (13), the time-dependent oscillatory solution!

We are **given** that $y(L/2, 0) = h$ (see Eq. 5).

Thus, putting $x = L/2$ and $t = 0$ in Eq. (13), we find

$$h = \frac{8h}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left[\frac{(2n+1)\pi}{2}\right]$$

$$=$$

Total energy E of the plucked string

Amazingly it turns out that we can evaluate the infinite series in Eq. (22) by using Eq. (13), the time-dependent oscillatory solution!

We are **given** that $y(L/2, 0) = h$ (see Eq. 5).

Thus, putting $x = L/2$ and $t = 0$ in Eq. (13), we find

$$\begin{aligned} h &= \frac{8h}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin\left[\frac{(2n+1)\pi}{2}\right] \\ &= \frac{8h}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n \times (-1)^n}{(2n+1)^2}. \end{aligned}$$

Total energy E of the plucked string

Lemma

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \tag{23}$$

Total energy E of the plucked string

Lemma

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}, \tag{23}$$

and the total energy, Eq. (22), becomes

$$E=\frac{2\rho h^2 c^2}{L}. \tag{24}$$

Total energy E of the plucked string

We can check this result from the initial conditions when

$$T = 0$$

and

$$V = \frac{1}{2}F\left[\int_0^{L/2}\left(\frac{2h}{L}\right)^2 dx + \int_{L/2}^{L}\left(\frac{-2h}{L}\right)^2 dx\right] =$$

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using Eq. (1.8). Thus

$$E|_{t=0} = 0 + \frac{2\rho h^2 c^2}{L}.$$

Fundamental frequency, overtones,...

• As a matter of fact, this worked example corresponds quite closely to a violin string plucked at its mid-point.
The fundamental frequency, or pitch, is

$$\frac{\pi c}{L}\frac{1}{2\pi} = \frac{c}{2L},$$

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Comments

The note heard by a listener depends on the amplitudes of the overtones; the note is not pure but the (relatively) rapid fall-off of the amplitudes means that the note is purer than that of many musical instruments, particularly the piano.
If the string had been bowed at some other point than its center, the amplitude of the overtones would have been different and thus tone would have been changed.

Outline

Fourier Transform

Series like Eqs. (4a) and (4b), and (perhaps!) Eqs. (13) and (16) are known as **Fourier Series** after the great French scientist and mathematician (Jean Baptiste) Joseph Fourier (1768-1830).

The methods used in this chapter are capable of extension in many different directions. The only one I want to draw attention to here is the following. We have seen Eq. (1.17) that

$$\Phi = A \exp[ik(x - ct)]$$

is a solution of the 1D wave equation for any value of the constant k.

Fourier Transform

So therefore is

$$A(k)\exp[ik(x-ct)]$$

for any function $A(k)$ and, also,

$$\int_{-\infty}^{\infty} A(k)\exp[ik(x-ct)]dk.$$

This leads/is related to Fourier Transforms or Fourier Analysis.[a]

[a] You might enjoy looking at “Fourier Analysis” by T.W.Kövner, CUP (1988).

Outline

Governing equation in Cartesian geometry

Consider a membrane, e.g. the surface of a drum. Let (x, y) denote position in the membrane and $z = z(x, y, t)$ be its transverse displacement. It can be easily shown that the governing equation for z is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right). \tag{25}$$

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$$z = X(x)Y(y)T(t).$$

Governing equation in cylindrical geometry

But for a drum it is more natural to use polar coordinates (r, θ) with

$$x = r\cos\theta, \quad y = r\sin\theta$$

when Eq. (25) becomes (for details see Appendix A at the end)

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$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right]. \qquad (26)$$

Normal modes in cylindrical geometry

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As in § (1.4) we seek separable solutions of the form

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$\Rightarrow$

$$\Theta\left[r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + R\frac{d^2\Theta}{d\theta^2} = -\frac{\omega^2}{c^2}r^2 R\Theta$$

$\Rightarrow$

Fourier decomposition in Θ-direction

$$\frac{1}{R}\left[r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right]+\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}+k^2r^2=0, \tag{27}$$

where

$$k=\frac{\omega}{c} \tag{28}$$

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- Suppose

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}=\text{const}:=-n^2$$

$\Rightarrow$

$$\Theta\propto e^{in\theta}.$$

Bessel's equation of order n

In practice, we must have

$$\Theta(\theta) = \Theta(\theta + 2\pi) \Rightarrow n \in N^{+}$$

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and the change of variable $\xi = kr$ gives

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$$r^2 R'' + rR' + (k^2 r^2 - n^2)R = 0,$$

and the change of variable $\xi = kr$ gives

$$\xi^2 \frac{\mathrm{d}^2 R}{\mathrm{d}\xi^2} + \xi \frac{\mathrm{d}R}{\mathrm{d}\xi} + (\xi^2 - n^2)R = 0. \tag{29}$$

This is known as **Bessel's equation of order *n***.

Standing modes

Rotational symmetry

We now consider only the case $n = 0 \Rightarrow$ no θ variation! The only solution of Eq. (29) that is bounded at $r = 0$ is

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Boundary condition

Assume, as with a drum, that the membrane is fixed at

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The University Of Sheffield.

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Assume, as with a drum, that the membrane is fixed at

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$$J_0(ka) = 0 \Rightarrow k = \frac{\lambda_m}{a}$$

where λ_m is the m-th root of $J_0(\xi)$.

Standing modes: Superposition

- Thus

$$z = A_m J_0\left(\frac{\lambda_m r}{a}\right) e^{i\lambda_m ct/a}$$

and the **general solution**, independent of θ, is

$$\mathbf{z = \sum_{n=1}^{\infty} A_m J_0\left(\frac{\lambda_m r}{a}\right) e^{i\lambda_m ct/a}}. \qquad (31)$$

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Exercise

Find $\{A_m\}$ by similar methods to those in § (2.2). (Note, there is an orthogonality relationship.)

Deriving 2D cylindrical wave equation

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\therefore \quad z_r =$$

$$z_\theta =$$

$$\therefore \quad z_x =$$

$$z_y =$$

Deriving 2D cylindrical wave equation

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\begin{aligned} \therefore \quad z_r &= \cos\theta z_x + \sin\theta z_y \\ z_\theta &= -r\sin\theta z_x + r\cos\theta z_y \end{aligned}$$

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$$\begin{aligned} \therefore \quad z_x &= \cos\theta z_r - \frac{\sin\theta}{r} z_\theta \\ z_y &= \sin\theta z_r + \frac{\cos\theta}{r} z_\theta \end{aligned}$$

Deriving 2D cylindrical wave equation

$$\therefore \quad z_{xx} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta z_r - \frac{\sin\theta}{r}z_\theta\right)$$

$$= \cos^2\theta z_{rr} + \frac{\cos\theta\sin\theta}{r^2}z_\theta - \frac{\cos\theta\sin\theta}{r}z_{r\theta}$$

$$+\frac{\sin^2\theta}{r}z_r - \frac{\sin\theta\cos\theta}{r}z_{r\theta}$$

$$+\frac{\sin\theta\cos\theta}{r^2}z_\theta + \frac{\sin^2\theta}{r^2}z_{\theta\theta}$$

$$= \cos^2\theta z_{rr} + \frac{\sin^2\theta}{r}z_r + \frac{\sin^2\theta}{r^2}z_{\theta\theta}$$

$$-\frac{2\cos\theta\sin\theta}{r^2}z_{r\theta} + \frac{2\cos\theta\sin\theta}{r^2}z_\theta .$$

Deriving 2D cylindrical wave equation

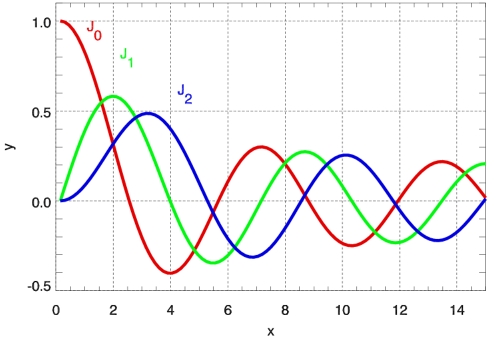
Likewise, after algebra:

$$\begin{aligned} z_{yy} &= \sin^2\theta z_{rr} + \frac{\cos^2\theta}{r} z_r + \frac{\cos^2\theta}{r^2} z_{\theta\theta} \\ &+ \frac{2\cos\theta\sin\theta}{r^2} z_{r\theta} - \frac{2\cos\theta\sin\theta}{r^2} z_\theta . \end{aligned} \quad (32)$$

Therefore

$$\begin{aligned} z_{xx} + z_{yy} &= z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta}, \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r z_r) + \frac{1}{r^2} z_{\theta\theta} . \end{aligned}$$

Bessel function J_n



- $y = J_n(x) \quad (n = 0, 1, 2, ..)$ is a solution of **Bessel's equation** of order n. This is (see Eq. (2.29) in Notes):

$$x^2 y'' + xy' + (x^2 - n^2)\, y = 0.$$

Bessel function J_n

- $J_n(x)$ is the **Bessel function of order** n defined precisely by the infinite series

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p x^{n+2p}}{2^{n+2p}\, p!\, (n+p)!}.$$

  This can be shown to satisfy Bessel's equation.
- The general solution of Bessel's equation of order n is unbounded as $x \to 0$. The most general solution that is bounded as $x \to 0$ is $y = AJ_n(x)$ where A is an arbitrary constant.
- As the sketches illustrate, $J_n(x)$ has an infinite number of zeros.

Bessel function J_n

- As the sketches illustrate, $J_n(x)$ has an infinite number of zeros.
- Let α_m be the mth zero of $J_0(x)$. To good approximation

$$\alpha_1 = 2.405, \quad \alpha_2 = 5.520, \quad \alpha_3 = 8.654$$

$$\alpha_m \approx \left(m - \frac{1}{4}\right)\pi \quad \text{for large } \; m$$