

Acoustic Waves

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With special thanx to



Outline

1 One-dimensional sound waves

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Mass conservation

- In general most phenomena associated with sound propagation can be described well by assuming that air (or the underlying fluid) has **uniform density** ρ_0 and **uniform pressure** p_0 in the undisturbed state, and that sound generation causes **small changes** in density ρ and pressure p , and a **small** velocity \mathbf{u} .

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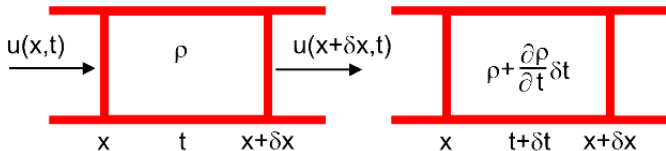
- We begin by assuming that the **motion is 1D**

\Rightarrow

$$\mathbf{u} = u(x, t)\mathbf{i}, \quad p = p(x, t), \quad \rho = \rho(x, t)$$

Mass conservation

- Now ρ and u are not independent since mass must be conserved.



Consider a small tube of length δx and cross-sectional area A .
 At **time t** the mass of fluid in the tube is

$$\rho(x, t) \delta x A$$

and at **time $t + \delta t$** it is

$$[\rho(x, t) + \delta \rho] \delta x A = \rho(x, t) \delta x A + \frac{\partial \rho(x, t)}{\partial t} \delta x \delta t A.$$

Mass conservation

The mass has **increased** by an amount

$$\rho_t \delta x \delta t A,$$

due to mass flowing **into** the tube: in time δt this is equal to

$$\begin{aligned} \{ \rho u |_{(x,t)} - \rho u |_{(x+\delta x,t)} \} \delta t A &= - \frac{\partial}{\partial x} (\rho u) \delta x \delta t A \\ &= - \rho \frac{\partial u}{\partial x} \delta x \delta t A - u \frac{\partial \rho}{\partial x} \delta x \delta t A. \end{aligned}$$

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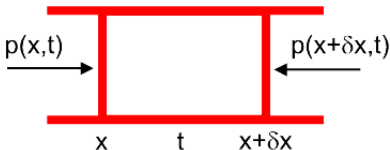
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to highest order. Thus the 1D equation of continuity becomes

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x}$$

Conservation of momentum

- Now we apply N2 to the fluid in the small tube. From Figure below the **force on the fluid** is



$$[p(x, t)A - p(x + \delta x, t)A] = -\frac{\partial p}{\partial x} \delta x A,$$

neglecting gravity and this is equal to

$$\rho \delta x A \frac{\partial u}{\partial t} \approx \rho_0 \frac{\partial u}{\partial t} \delta x A$$

by N2. Thus

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}.$$

Wave equation in 1D

- Now, by Eq. (1),

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x} = \left(\frac{dp}{d\rho} \right)_{\rho=\rho_0} \frac{\partial \rho}{\partial x} := c^2 \frac{\partial \rho}{\partial x},$$

where

$$c^2 = (\partial p / \partial \rho). \quad (4)$$

Thus, from Eq. (2),

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 \frac{\partial^2 u}{\partial x \partial t} = -\rho_0 \frac{\partial}{\partial x} \left(-\frac{1}{\rho_0} c^2 \frac{\partial \rho}{\partial x} \right), \Rightarrow$$
$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \frac{\partial^2 \rho}{\partial x^2}.$$

Likewise, the same equation holds for u (and p). Thus

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \& \quad \frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2}. \quad (5)$$

Sound in air

- For sound in gases, the appropriate relationship Eq. (1) between p and ρ is the **adiabatic law***

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma \Rightarrow \frac{\partial p}{\partial \rho} =$$

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*NB: No heat exchange \Rightarrow change occurs quickly. Newton derived Eq. (5), but assumed that p and ρ were related by the **isothermal** law (Boyle's Law)

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}$$

\Rightarrow

$$c = \left(\frac{p_0}{\rho_0} \right)^{1/2} \approx$$

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$$\frac{p}{\rho} = \frac{p_0}{\rho_0}$$

\Rightarrow

$$c = \left(\frac{p_0}{\rho_0} \right)^{1/2} \approx 280 \text{ m s}^{-1}$$

Correct expression due to Laplace!

Sound in air

By Eq. (4), the velocity of sound in air is

$$c = \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2} \approx \quad (6)$$

at standard temperature and pressure, e.g. $T=288$ K; $\gamma = 1.4$,
 $p_0 \approx 1.013 \times 10^5$ Nm⁻², $\rho_0 \approx 1.293$ kgm⁻³.

Sound in air

By Eq. (4), the velocity of sound in air is

$$c = \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2} \approx 330 \text{ m s}^{-1} \quad (6)$$

at standard temperature and pressure, e.g. $T=288 \text{ K}$; $\gamma = 1.4$,
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Eq. (6) agrees well with experiments.

Sound in water

The same analysis applies to water where experiments show that

$$(p - p_0) = \kappa(\rho - \rho_0)/\rho_0$$

\Rightarrow

$$\frac{dp}{d\rho} = \frac{\kappa}{\rho_0}$$

\Rightarrow

$$c \approx \dots?$$

$$(\kappa \approx 2.045 \times 10^9 \text{ kgm}^{-1}\text{s}^{-2}, \rho_0 \approx 10^3 \text{ kg m}^{-3})$$

Sound in water

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$$(p - p_0) = \kappa(\rho - \rho_0)/\rho_0$$

\Rightarrow

$$\frac{dp}{d\rho} = \frac{\kappa}{\rho_0}$$

\Rightarrow

$$c \approx 1.430 \times 10^3 \text{ m s}^{-1}$$

$$(\kappa \approx 2.045 \times 10^9 \text{ kgm}^{-1}\text{s}^{-2}, \rho_0 \approx 10^3 \text{ kg m}^{-3})$$

- One dimensional sound waves can be treated by the same mathematical methods as (one-dimensional) waves on strings. Note, however, that in strings the motion is **transverse**; in sound waves it is **longitudinal**.

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3D fluid

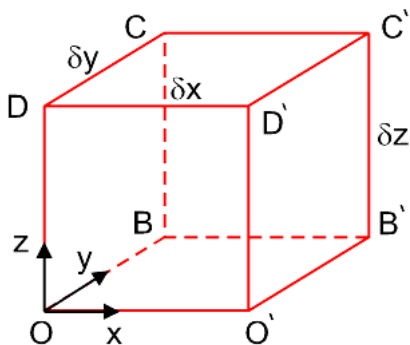
- In reality, sound propagates in 3D (although the 1D result can be applied approximately to e.g. pipes). We now assume the fluid velocity \mathbf{u} is still **small** but with three non-zero components:

$$\mathbf{u} = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}. \quad (7)$$

We must again ensure that mass is conserved - the resulting equation, viz. **Eq. (8)**, is known as the **equation of continuity**.

- Consider a small cuboid with sides parallel to the axes, and of lengths $\delta x, \delta y, \delta z$.

3D fluid element $\delta x \delta y \delta z$



As above, the **increase** in the mass within this cuboid between times t and δt is

$$\frac{\partial \rho}{\partial t} \delta x \delta y \delta z \delta t.$$

Mass conservation in 3D

This is equal to the mass flowing **into** the cuboid, i.e.

$$\begin{aligned} & \{\rho u|_{OBCD} - \rho u|_{O'B'C'D'}\} \delta y \delta z \delta t \\ + & \{\rho v|_{OO'D'D} - \rho v|_{BB'C'C}\} \delta z \delta x \delta t \\ + & \{\rho w|_{OO'B'B} - \rho w|_{DD'C'C}\} \delta x \delta y \delta t \\ \approx & - \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] \delta x \delta y \delta z \delta t. \end{aligned}$$

Thus

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 \approx & - \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] \delta x \delta y \delta z \delta t.
 \end{aligned}$$

Thus

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \quad (8)$$

Eq. (8) is **exact** since no assumption of smallness has yet been made. This is the **equation of continuity** for any fluid.

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Linear mass conservation in 3D

- When $\rho - \rho_0$ and u, v, w are small, Eq. (8) becomes

$$\frac{\partial \rho}{\partial t} + \rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \underbrace{(\rho - \rho_0) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_{+u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}} = 0.$$

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$$\underbrace{+ u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}}_{\text{2nd order}} = 0.$$

Thus

$$\frac{\partial \rho}{\partial t} = -\rho_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \quad (9)$$

replaces Eq. (2) for acoustic waves.

Linear conservation of momentum in 3D

- N2 gives three equations like Eq. (3)¹ \Rightarrow :

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y}, \quad \frac{\partial w}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} \quad (10)$$

or,

¹The LHS of Eq. (10) - and Eq. (3) earlier - involve an assumption of u, v, w being small. See § (4.1)

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or, using Eq. (4):

$$\frac{\partial u}{\partial t} = -\frac{c^2}{\rho_0} \frac{\partial \rho}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{c^2}{\rho_0} \frac{\partial \rho}{\partial y}, \quad \frac{\partial w}{\partial t} = -\frac{c^2}{\rho_0} \frac{\partial \rho}{\partial z} \quad (11)$$

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Linear wave equation in 3D

Thus, from Eq. (9)

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0 \left(\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial^2 w}{\partial z \partial t} \right)$$
$$=$$

Linear wave equation in 3D

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Hence

$$\frac{\partial^2 \rho}{\partial t^2} = c^2 \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} \right). \quad (12)$$

This is the **3D wave equation** with **speed c** .

Velocity potential

- In most circumstances $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad w = \frac{\partial\phi}{\partial z}, \quad (13)$$

and ϕ is the **velocity potential**. It can be shown that ϕ also satisfies the same equation (so therefore do u, v, w):

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and ϕ is the **velocity potential**. It can be shown that ϕ also satisfies the same equation (so therefore do u, v, w):

$$\frac{\partial^2\phi}{\partial t^2} = c^2 \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \right). \quad (14)$$