Surface Waves in Fluids

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With special thanx to .....!

Outline

1. The governing equations
   - Basic concepts
   - The governing equations for 1D water waves
   - Monochromatic surface waves

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   - Group velocity I
   - Group velocity II

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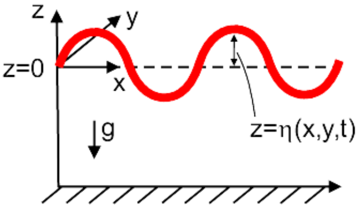
2. Energy
   - Equipartition
   - Group velocity I
   - Group velocity II
3. Practical concepts
   - The Doppler effect
   - Particle paths in surface waves
   - Appendix A

Outline

The University Of Sheffield.

Assumptions: Linearity

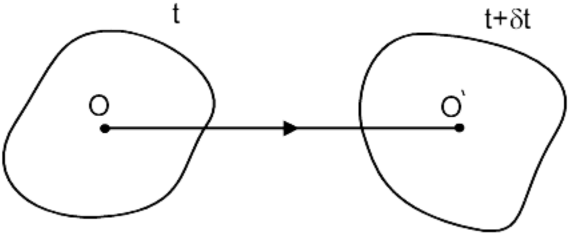
• We consider waves on the surface of liquids, e.g. waves on the sea or a lake or a river. These can be generated by the wind, by a moving boat... One key factor: if the surface is displaced from its equilibrium position $z = 0$ to $z = \eta(x, y, t)$, gravity will tend to restore the surface to its equilibrium position.



• In practice we can assume that the disturbance is small (i.e. the amplitude, e.g. $\sup|\eta|$, is much less than the wavelength) $\Rightarrow$ linear theory.

Assumptions: Incompressible fluids

• A further simplification is that for most liquids the equation of continuity Eq. (3.8) can be considerably simplified because liquids are difficult to compress
$\Rightarrow$ volume of small piece of liquid is unchanged as it moves
$\Rightarrow$ density is unchanged (since mass = density $\times$ volume and mass is conserved).



$$\vec{OO'} = \mathbf{u}(\mathbf{x}, t)\delta t$$

Assumptions: Incompressible fluids

Consider a small volume of liquid of density ρ. Suppose it is at $\mathbf{x}$ at time t ; in a small interval of time δt, the volume will have moved from

$$\mathbf{x} \quad \rightarrow \quad \mathbf{x} + \mathbf{u}(\mathbf{x}, t)\delta t,$$

so the density will have changed from

$$\rho(\mathbf{x}, t) \quad \rightarrow \quad \rho(\mathbf{x} + \mathbf{u}\delta t, t + \delta t).$$

By hypothesis, these are the same. But...

$$\begin{aligned}\rho(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) &= \rho(x + u\delta t, y + v\delta t, z + w\delta t, t + \delta t) \\ &\approx\end{aligned}$$

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$$\begin{aligned} \rho(\mathbf{x} + \mathbf{u}\delta t, t + \delta t) &= \rho(x + u\delta t, y + v\delta t, z + w\delta t, t + \delta t) \\ &\approx \rho(x, y, z, t) \\ &\quad + (u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t})\delta t. \end{aligned}$$

Assumptions: Incompressible fluids

Thus

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0, \tag{1}$$

where D/Dt is the operator defined by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right). \tag{2}$$

D/Dt applied to any function of $(\mathbf{x}, t)$ measures rate of change when moving with the liquid (or fluid).

Assumptions: Incompressible fluids

From mass conservation Eq. (3.8) we have

$$\frac{\partial \rho}{\partial t} + \left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0,$$

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so from Eq. (1) $\Rightarrow$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (3)$$

which is the equation of continuity for an incompressible fluid or liquid.

Assumptions: Incompressible fluids

• The reasoning applied to $\rho(\mathbf{x}, t)$ above can also be applied to $\mathbf{u}(\mathbf{x}, t)$ (i.e. velocity).
The rate of change of the velocity of the piece of fluid, i.e. its acceleration, is

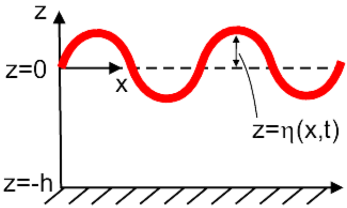
$$\frac{D}{Dt}(u, v, w) = \frac{\partial}{\partial t}(u, v, w) + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right)(u, v, w).$$

But our assumption that the disturbance is small $\Rightarrow$ second term is small $\Rightarrow$

$$\text{acceleration} \approx \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right).$$

(This has already been used in Eqs (3.3) and (3.10).)

Assumptions: Free surface



• We consider only cases where the disturbance of the free surface is independent of $y \Rightarrow z = \eta(x, t)$ is the disturbance. We therefore assume that

$$\mathbf{u} = u(x, z, t)\mathbf{i} + w(x, z, t)\mathbf{k} \tag{4}$$

Then Eq. (3) $\Rightarrow$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{5}$$

Momentum equation

Recalling N2 $\Rightarrow$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \quad \rho \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \rho g \tag{6}$$

where ρ can be regarded as constant*, and the term ρg represents the weight $\rho g\,\delta V$ acting vertically downwards.

*This is an extension of Eq. (1) - we assume ρ is an absolute constant, independent of both $\mathbf{x}$ and t. In the ocean, ρ does vary (slightly) with height, but not enough to affect the analysis of surface waves.

Velocity potential

- As in § (3.3), it can be shown that, in most circumstances, there is a velocity potential, ϕ such that Eq. (3.13) holds[1]. In the present case $\phi = \phi(x, z, t)$ and

$$u = \frac{\partial \phi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z} \tag{7}$$

Then Eq. (5) becomes

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This is the 2D form of .... equation and is the PDE that must be solved. NB: Surface waves are not governed by the wave equation!

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Boundary conditions

• The special features of surface waves arise because of the boundary conditions. There will be three in the problems we consider:

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3. the pressure at the free surface must be continuous and since the density of air is much less than that of water, we can assume the air pressure is constant p_0.

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$$\frac{D}{Dt}\{z - \eta(x,t)\} = 0 \quad \text{at} \quad z = \eta$$

$\Rightarrow$

$$w - \frac{\partial \eta}{\partial t} - \underbrace{u\frac{\partial \eta}{\partial x}}_{\approx \text{ small}} = 0 \quad \text{at} \quad z = \eta.$$

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$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0. \tag{10}$$

Boundary conditions

- (3) : Eq. (6) $\Rightarrow$

$$\frac{\partial}{\partial x}\left\{\frac{p-p_0}{\rho}+\dot{\phi}\right\}=\frac{\partial}{\partial z}\left\{\frac{p-p_0}{\rho}+\dot{\phi}+gz\right\}=0$$

$\Rightarrow$

$$\frac{p-p_0}{\rho}+\frac{\partial \phi}{\partial t}+gz \quad \text{depends only on} \quad t.$$

However we can incorporate this function of t by adding it to ϕ. This has no effect on **u** by Eq. (7). Since $p = p_0$ at $z = \eta$, and we are linearising, we can apply this condition at $z = 0$ as far as ϕ is concerned. Thus from (3) $\Rightarrow$

$$\frac{\partial \phi}{\partial t}+g\eta=0 \quad \text{at} \quad z=0. \tag{11}$$

Monochromatic waves

- Monochromatic $\Rightarrow$ single wave number k, single (angular) frequency ω. We assume the free surface is given by

$$\begin{aligned} \eta = \eta_0 \sin(kx - \omega t) &= \eta_0 \sin k(x - ct) \\ \omega &= kc \end{aligned} \qquad (12)$$

Eq. (12b) has already been used several times, e.g. Eq. (2.28).

Note:

We could also work with the complex form Eq. (1.27), viz.

$$\eta = \eta_0^* \mathrm{e}^{i(kx - \omega t)}.$$

Applying BCs

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$$\phi = A\cosh k(z+h)\cos(kx - \omega t) \tag{15}$$

Itermezzo: Using exponential notation

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$$f = \gamma e^{kz} + \delta e^{-kz}.$$

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Thus

$$\begin{aligned} f &= \gamma \mathrm{e}^{kz} + \gamma \mathrm{e}^{-2kh} \mathrm{e}^{-kz} = \gamma \mathrm{e}^{-kh} \mathrm{e}^{k(z+h)} + \gamma \mathrm{e}^{-kh} \mathrm{e}^{-k(z+h)} \\ &= 2\gamma \mathrm{e}^{-kh} \cosh k(z+h) = A \cosh k(z+h) \end{aligned}$$

with

$$A = 2\gamma \mathrm{e}^{-kh}.]$$

DR and solution

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From Eq. (11) $\Rightarrow$

$$\omega A\cosh kh\sin(kx-\omega t) + g\eta_0\sin(kx-\omega t) = 0$$

$\Rightarrow$

$$-g\eta_0 = \omega A\cosh kh \qquad \text{(B)}$$

Then (A)/(B) $\Rightarrow$

DR and solution

$$\frac{\omega}{g} = \frac{k}{\omega} \tanh kh \quad \Rightarrow$$

$$\omega^2 = gk \tanh kh, \qquad c^2 = \frac{g}{k} \tanh kh \tag{16}$$

and

$$\phi = -\frac{g\eta_0}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \cos(kx - \omega t). \tag{17}$$

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Thus waves with different wavelengths travel at different speeds c, where

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This phenomenon is known as dispersion.

Two practical limits

We note two special cases:

$$\text{Deep water} \quad h \to \infty, \tag{19a}$$

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Note:

- Shallow water waves are not dispersive.
- This is a progressive wave, but standing waves can be dealt with similarly - see S4 Q3.

Outline

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$$\left(\int_0^{\eta} \rho g z dz\right) \delta A = \frac{1}{2}\rho g \eta^2 \delta A.$$

Thus the PE in a wavelength per unit width in the direction of $0y$ is V_*, where using Eq. (12) results in

$$V_* = \frac{1}{2}\rho g \eta_o^2 \int_0^{2\pi/k} \sin^2 k(x - ct) dx$$

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This is

$$\frac{1}{2}\rho g \eta_0^2 \cdot \frac{\pi}{k} = \frac{1}{4}\rho g \eta_0^2 \lambda,$$

where $\lambda = \frac{2\pi}{k}$ is the wavelength. Thus the potential energy density per unit area of water surface is $V_\rho = V_*/\lambda$.

$$V_\rho = \frac{1}{4}\rho g \eta_0^2 \qquad (20a)$$

Kinetic energy

• Likewise the KE in a wavelength per unit width in the direction of $0y$ is T_*, where

$$T_* = \frac{1}{2}\rho \int_{-h}^{0} dz \int_{0}^{2\pi/k} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right] dx$$

Here

$$\left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{g^2 \eta_0^2 k^2}{\omega^2 \cosh^2 kh} \cosh^2 k(z+h) \sin^2 k(x-ct)$$

and

$$\left(\frac{\partial \phi}{\partial z}\right)^2 = \frac{g^2 \eta_0^2 k^2}{\omega^2 \cosh^2 kh} \sinh^2 k(z+h) \cos^2 k(x-ct)$$

Kinetic energy

Since (as with V_*)

$$\int_0^{2\pi/k} \sin^2 k(x-ct)dx = \int_0^{2\pi/k} \cos^2 k(x-ct)dx = \frac{\pi}{k},$$

we find

$$T_* = \frac{\pi}{2}\frac{\rho g^2 \eta_0^2 k}{\omega^2 \cosh^2 kh}\int_{-h}^{0} \cosh 2k(z+h)dz,$$

(since $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$). Thus

$$T_* = \frac{\pi}{4}\frac{\rho g^2 \eta_0^2}{\omega^2 \cosh^2 kh}\cdot \sinh 2kh = \frac{\pi}{2}\frac{\rho g^2 \eta_0^2}{\omega^2}\tanh kh$$

(since $\sinh 2\theta = 2\sinh\theta\cosh\theta$).

Kinetic energy

$\Rightarrow$ from Eq. (16),

$$T_* = \frac{1}{2}\rho g \eta_0^2 \frac{\pi}{k} = \frac{1}{4}\rho g \eta_0^2 \lambda = V_*.$$

$\Rightarrow$ the kinetic energy density per unit area of the water surface is $T_\rho = T_*/\lambda$

$$T_\rho = \frac{1}{4}\rho g \eta_0^2. \quad (20b)$$

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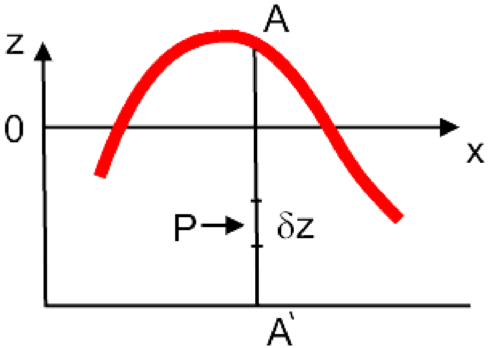
Note:

The kinetic energy density per unit area (T_ρ) of the water surface wave is equal to the potential energy density per unit area (V_ρ).

Work by surface waves



• We now calculate the rate at which work is being done by the fluid on the left of AA' on the fluid on the right. The force per unit width in the direction of $0y$ is $p\,\delta z$ so its rate of working P_* (for power) per unit width is given by

Work by surface waves

$$P_* = \int_{-h}^{0} pu\,dz = \int_{-h}^{0} \left(p_0 - \rho \frac{\partial \phi}{\partial t} - gz \right) \frac{\partial \phi}{\partial x} dz,$$

since $p = p_0 - \rho \frac{\partial \phi}{\partial t} - gz$ from derivation of Eq. (11).
It is sufficient for our purposes to calculate the mean of P_* over one period.

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since $p = p_0 - \rho\frac{\partial\phi}{\partial t} - gz$ from derivation of Eq. (11).
It is sufficient for our purposes to calculate the mean of P_* over one period.

Since the mean of $\sin k(x - ct)$ is 0, and the mean of $\sin^2 k(x - ct) = \frac{1}{2}$, we let P be the mean of P_* and find:

$$P = \frac{\rho g^2 \eta_0^2 k}{2\omega \cosh^2 kh} \int_{-h}^{0} \cosh^2 k(z + h) dz$$

after some algebra (exercise for student).

Work by surface waves

Since $c = \omega/k$ and $\cosh^2\theta = \frac{1}{2}(1+\cosh 2\theta)$, we find

$$\begin{aligned} P &= \frac{\rho g^2 \eta_0^2}{4c\cosh^2 kh}\left[h + \frac{\sinh 2kh}{2k}\right] \\ &= \frac{\rho g^2 \eta_0^2}{8kc}\left[1 + \frac{2kh}{\sinh 2kh}\right] 2\tanh kh \end{aligned}$$

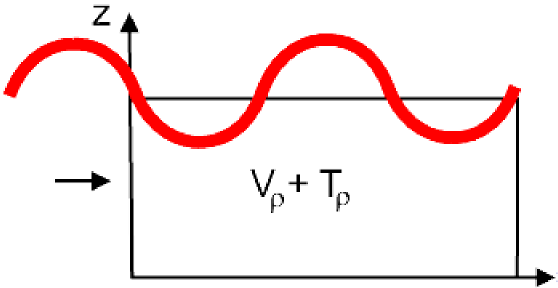
since $\sinh 2\theta = 2\sinh\theta\cosh\theta$. Thus, using Eq. (16),

$$P = \frac{1}{4}\rho g \eta_0^2 c\left[1 + \frac{2kh}{\sinh 2kh}\right]. \qquad (21)$$

Energy propagation → group velocity

• There is an interesting and important consequence of Eqs. (20a), (20b) and (21) which can be extended to many sorts of waves leading to the concept of group velocity.

As a consequence of the passage of the waves, energy is being transmitted from left to right with a (mean) speed U to be determined.



Energy propagation → group velocity

In a time τ, this results in new energy per unit width equal to $(V_\rho + T_\rho)U\tau$, and this must be equal to $P\tau$, the work done.

$\Rightarrow$

$$U = P/(V_\rho + T_\rho) = P/(\frac{1}{2}\rho g\eta_0^2),$$

i.e.

$$U = c_g = \frac{1}{2}c\left[1 + \frac{2kh}{\sinh 2kh}\right] \tag{22}$$

where

c_g is known as the group velocity for reasons that will be discussed later.

DR $\rightarrow$ group velocity

- From the first of Eq. (16), we have

$$2\omega \frac{\mathrm{d}\omega}{\mathrm{d}k} = g \tanh kh + \frac{gkh}{\cosh^2 kh}$$

(since $\frac{\mathrm{d}}{\mathrm{d}\theta}(\tanh\theta) = \mathrm{sech}^2\theta = \frac{1}{\cosh^2\theta}$). Thus

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{g\tanh kh}{2\omega}\left[1 + \frac{kh}{\tanh kh \cosh^2 kh}\right] = \frac{kc^2}{2\omega}\left[1 + \frac{2kh}{\sinh 2kh}\right],$$

i.e. see Eqs. (18) and (22) $\Rightarrow$

Energy propagation, DR → group velocity

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{23}$$

Eq. (23) is the general definition of group velocity.

Note:

Energy propagation, DR $\rightarrow$ group velocity

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Eq. (23) is the general definition of group velocity.

Note:

Recall, $\omega = kc$ so that, when c is independent of k, as for waves on strings and sound waves, i.e. when the waves are non-dispersive, Eq. (23) gives

$$c_g = c \tag{24}$$

i.e. the group velocity c_g is equal to c, the phase velocity.

Two practical limits

• Finally, we record the results for the two special cases considered in Eqs. (19a)-(19b)

Deep water $h \to \infty \Rightarrow$

(25a)

Shallow water $kh \ll 1 \Rightarrow$

(25b)

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Shallow water $kh \ll 1 \Rightarrow$

$$\omega^2 \approx gk^2 h, \quad c^2 = gh, \quad c_g = \sqrt{gh} = c \tag{25b}$$

Properties of group velocity: Superposition

• Energy is (often and at an average over time) transported at the group velocity c_g. This applies to many sorts of wave.

• There are other important properties of c_g. Consider the superposition of two waves like Eq. (12) in the case when the amplitudes are equal but the waves numbers and frequencies are slightly different. We have

$$\begin{aligned}\eta &= \eta_0 \sin(kx - \omega t) + \eta_0 \sin[(k + \delta k)x - (\omega + \delta\omega)t] \\ &= \end{aligned}$$

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$$\begin{aligned}\eta &= \eta_0 \sin(kx - \omega t) + \eta_0 \sin\left[(k + \delta k)x - (\omega + \delta\omega)t\right] \\ &= 2\eta_0 \sin\left[(k + \frac{1}{2}\delta k)x - (\omega + \frac{1}{2}\delta\omega)t\right] \\ &\qquad \times \cos\left[\frac{1}{2}\delta k(x - \frac{\delta\omega}{\delta k}t)\right]\end{aligned}$$

$\Rightarrow$

Properties of group velocity: Superposition

$$\eta \approx 2\eta_0 \cos\left[\frac{1}{2}\delta k(x - c_g t)\right] \sin[kx - \omega t] \qquad \left(c_g \approx \frac{\delta\omega}{\delta k}\right) \quad (26)$$

The combined displacement can be thought of as the original wave but with an amplitude that gradually changes between $\pm 2\eta_0$ over a distance $\pi/(\frac{1}{2}\delta k) = 2\pi/(\delta k)$.

The covering surface will be a series of groups of waves, separated by essentially smooth water where

$$\cos[\frac{1}{2}\delta k(x - c_g t)] \approx 0.$$

The groups are travelling at speed c_g, whereas the individual waves within each group are travelling at speed c. See sketch in Appendix A.

Properties of group velocity: Beats

NB In passing, suppose η is density or velocity potential in sound waves, where $c_g = c$. Then Eq. (26) becomes

$$\eta \approx 2\eta_0 \cos\left[\frac{1}{2}(\delta k x - \delta\omega t)\right] \sin\left[kx - \omega t\right],$$

so that the wave has a fluctuating intensity known as beats ; the beat frequency is $\delta\omega$ (see Appendix A).

Properties of group velocity: Wave packet

• We can develop the above analysis to consider a wave packet. As noted in § (2.5i), we can generalise to consider the disturbance $\eta(x,t)$, where

$$\eta(x,t) = \int_{-\infty}^{\infty} A(k)\mathrm{e}^{i(kx-\omega t)}\,dk,$$

and the real part of this is eventually to be taken. Here we shall consider the special case when

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and the real part of this is eventually to be taken. Here we shall consider the special case when

$$A(k) = A\mathrm{e}^{-d^2(k-k_0)^2},$$

where A, d, k_0 are constants. This gives the Gaussian wave packet

$$\eta(x,t) = A\int_{-\infty}^{\infty} \mathrm{e}^{-d^2(k-k_0)^2}\mathrm{e}^{i(kx-\omega t)}dk, \qquad (27)$$

Properties of group velocity: Wave packet

The dominant contribution comes from values of k near k_0 because of the nature of $\mathrm{e}^{-d^2(k-k_0)^2}$. We write

$$\omega = \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k-k_0) + \ldots \quad = \quad \omega_0 + c_g(k-k_0) + \ldots$$

and neglect terms of higher order to obtain

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$$\eta(x,t) = A\mathrm{e}^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \mathrm{e}^{-d^2(k-k_0)^2 + i(k-k_0)(x-c_g t)} dk$$

Substitute $d(k-k_0) = \xi$ to obtain (after some algebra):

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$$\eta(x,t) = A\mathrm{e}^{i(k_0 x - \omega_0 t} \int_{-\infty}^{\infty} \mathrm{e}^{-\{\xi - \frac{i}{2d}(x-c_g t)\}^2} \mathrm{e}^{-\frac{(x-c_g t)^2}{4d^2}} \frac{d\xi}{d}$$

Properties of group velocity: Wave packet

Now substitute $\zeta = \xi - \frac{i}{2d}(x - c_g t)$ to get

$$\eta(x, t) = \qquad (28)$$

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$$\eta(x,t) = \frac{A}{d} \mathrm{e}^{-(x-c_g t)^2/(4d^2)} \mathrm{e}^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} \mathrm{e}^{-\zeta^2} d\zeta$$

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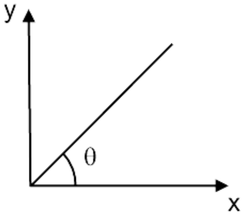
Now

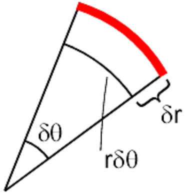
$$\int_{-\infty}^{\infty} \mathrm{e}^{-\zeta^2} d\zeta = \sqrt{\pi} \qquad (28)$$

Proof of Eq. (28) is via a clever trick!

Properties of group velocity: Wave packet

Proof:





$$I = \int_{-\infty}^{\infty} e^{-x^2} = \int_{-\infty}^{\infty} e^{-y^2}$$

Properties of group velocity: Wave packet

$$
\begin{aligned}
\therefore \quad I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{e}^{-y^2}\,dxdy \\
&= \int_0^{2\pi} d\theta \int_0^{\infty} \mathrm{e}^{-r^2}\,rdr \\
&=
\end{aligned}
$$

Properties of group velocity: Wave packet

$$\therefore \quad I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{e}^{-y^2}\,dxdy$$

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$$= 2\pi \int_0^{\infty} r\mathrm{e}^{-r^2} =$$

Properties of group velocity: Wave packet

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$$= \int_0^{2\pi} d\theta \int_0^{\infty} \mathrm{e}^{-r^2}\,rdr$$

$$= 2\pi \int_0^{\infty} r\mathrm{e}^{-r^2} = 2\pi\left[-\frac{1}{2}\mathrm{e}^{-r^2}\right]_0^{\infty} =$$

Properties of group velocity: Wave packet

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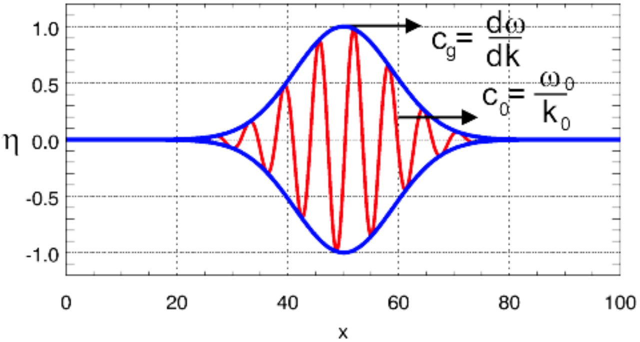
$$= 2\pi \int_0^{\infty} r\mathrm{e}^{-r^2} = 2\pi\left[-\frac{1}{2}\mathrm{e}^{-r^2}\right]_0^{\infty} = \pi$$

$$\therefore \quad I = \sqrt{\pi}$$

Properties of group velocity: Wave packet

Thus

$$\eta(x,t) = \frac{A\sqrt{\pi}}{d} e^{-\frac{(x-c_g t)^2}{4d^2}} e^{i(k_0 x - \omega_0 t)} \tag{29}$$



Outline

1. The governing equations
   - Basic concepts
   - The governing equations for 1D water waves
   - Monochromatic surface waves
2. Energy
   - Equipartition
   - Group velocity I
   - Group velocity II
3. Practical concepts
   - The Doppler effect
   - Particle paths in surface waves
   - Appendix A

Waves with moving source (or observer)

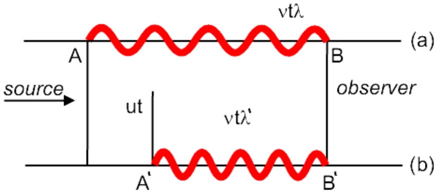
• It is convenient here[2] to consider another general phenomenon connected with waves, namely the changes in frequency of waves sent out by a moving source and perceived by a stationary observer.

Consider sound waves for sound waves for definiteness. We shall work in terms of the actual frequency ν and the wavelength λ, where

$$\nu = \frac{\omega}{2\pi}, \qquad \lambda = \frac{2\pi}{k}, \qquad c = \nu\lambda \tag{30}$$

[2] But not logical since it is more relevant to sound waves and radio waves than to surface waves than to surface waves on water!

Waves with moving source (or observer)



- (a) Waves when source is stationary;
- (b) Waves when source is moving.

In a time *t* the source emits νt waves. For a stationary source these occupy a length $\nu t\lambda$, whereas, for a source moving with speed *u* towards the observer, the wavelength changes to λ' and the νt waves occupy a distance $\nu t\lambda'$.

Waves with moving source (or observer)

Thus

$$\nu t\lambda = \nu t\lambda' + ut \quad \Rightarrow \quad \lambda' = \lambda - \frac{u}{\nu}$$

$$\lambda' = \lambda(1 - \frac{u}{c}) \tag{31}$$

As a result the observer measures the frequency of the waves as ν' where $\nu'\lambda' = c = \nu\lambda$. Thus

$$\nu' = \frac{\nu c}{c - u} \tag{32}$$

This phenomenon is called the Doppler shift.

Waves with moving source (or observer)

Example

An observer at rest notices that the frequency of the sound waves from a car appears to drop from 281 Hz to 257 Hz as the car passes. Given that the speed of sound is 330 ms^{-1}, estimate the speed of the car.

From Eq. (32) $\Rightarrow$

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From Eq. (32) $\Rightarrow$

$$281 = \frac{\nu}{1-\frac{u}{c}}, \quad 257 = \frac{\nu}{1+\frac{u}{c}} \Rightarrow \quad \frac{281}{257} = \frac{1+\frac{u}{c}}{1-\frac{u}{c}}$$

$\Rightarrow$

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$\Rightarrow$

$$\frac{u}{c} = \frac{24}{538} \quad \Rightarrow \quad u \approx$$

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$\Rightarrow$

$$\frac{u}{c} = \frac{24}{538} \quad \Rightarrow \quad u \approx 14.7\,\mathrm{m\ s}^{-1} \quad (\text{About} \quad 33\,\mathrm{mph})$$

How do particles move in surface waves?

Consider a particle whose equilibrium position is (x_0, z_0). Suppose its position at time t is $(x_0 + X(t), z_0 + Z(t))$, where the time means of X and Z will be chosen to be zero. Then

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left.\frac{\partial\phi}{\partial x}\right|_{(x_0+X, z_0+Z)} \approx \left.\frac{\partial\phi}{\partial x}\right|_{(x_0, z_0)}$$

$$= \tag{33}$$

using Eq. (17).

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$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \left.\frac{\partial \phi}{\partial z}\right|_{(x_0+X, z_0+Z)} \approx \left.\frac{\partial \phi}{\partial z}\right|_{(x_0, z_0)}$$

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$$
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&= -\frac{kg\eta_0 \sinh k(z_0 + h)}{\omega \cosh kh} \cos(kx_0 - \omega t)
\end{aligned}
$$

using Eq. (17). Integrating and ensuring zero time means:

How do particles move in surface waves?

$$
\begin{aligned}
X &= \\
Z &=
\end{aligned}
\qquad (34)
$$

using the first of Eq. (16).

How do particles move in surface waves?

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\begin{aligned}
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It follows on eliminating $\cos(kx_0 - \omega t)$ and $\sin(kx_0 - \omega t)$ that

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$$\frac{X^2}{a^2} + \frac{Z^2}{b^2} = 1 \quad \text{where} \left\{ \begin{array}{lcl} a & = & \dfrac{\eta_0 \cosh k(z_0+h)}{\sinh kh}, \\ b & = & \dfrac{\eta_0 \sinh k(z_0+h)}{\sinh kh} \end{array} \right. \qquad (35)$$

Summary:

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Summary:

1. Thus the particle paths are ellipses.

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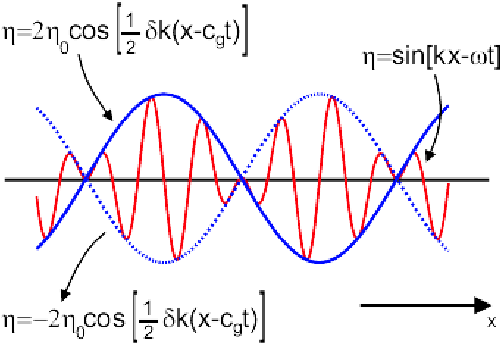
Summary:

1. Thus the particle paths are ellipses.
2. As $z_0 \to -h$, $b \to 0$, $a \to \eta_0 / \sinh kh \Rightarrow$ rectilinear motion in direction of $0x$.

Group velocity



Sketch for Eq. (4.26).