(**1**) Use Eq. (1.11) to write down the general solution of

$$\phi_{xx} = \phi_{yy}.$$

Find the solution of this equation on $-\infty < x < \infty$, $y \geq 0$ in the following three cases:

$$\begin{aligned}
(i)\ \ \phi(x,0) &= 0, \quad \frac{\partial \phi}{\partial y}(x,0) = 4x; \\
(ii)\ \ \phi(x,0) &= \cos kx, \quad \frac{\partial \phi}{\partial y}(x,0) = k\sin kx; \quad (k \text{ constant}) \\
(iii)\ \ \phi(x,0) &= 0, \quad \frac{\partial \phi}{\partial y}(x,0) = k\sin kx; \quad (k \text{ constant}).
\end{aligned}$$

(**2**) The transverse displacement $y(x,t)$ of a string satisfies

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where c is a constant. Write down (no working required) the general solution of this equation.

Given that the string is everywhere at rest at $t = 0$ and that

$$y(x,0) = \begin{cases} 0, & (-\infty < x < -a); \\ (a^2 - x^2), & (-a \leq x \leq a); \\ 0, & (a < x < \infty), \end{cases}$$

find $y(x,t)$ for all x and all $t \geq 0$.

Sketch the graphs of y against x when (i) $ct = 0$, (ii) $ct = \frac{1}{2}a$, (iii) $ct = 2a$.

(**3**) What conditions (if any) must the constants A, $c(> 0)$ and k satisfy for $u = A\cos[k(x - ct)]$ to be a solution to each of the following two PDEs (Partial Differential Equations)?

$$(i)\ \ u_t + au_x = 0, \qquad (ii)\ \ u_{tt} = au_{xx} - bu,$$

where a, b are positive constants.

(NB There are two separate problems here.)

For (ii) (the *Klein-Gordon equation*), sketch a graph of c against k.

(**4**) Consider a wave on an infinite string ($-\infty < x < \infty$) of uniform line density ρ, uniform tension F and wave speed c (where $c^2 = F/\rho$) of the form $y = f(x - ct)$. Assuming that $f'(u) \to 0$ as $|u| \to \infty$ sufficiently rapidly for the relevant integral(s) to converge, use Eqs. (1.7) and (1.8) to show that the kinetic energy T and the potential energy V are equal.

Verify that the same conclusion holds if $y = g(x + ct)$ (provided, again, that $g'(u) \to 0$ as $|u| \to \infty$ sufficiently rapidly). What can you say when $y = f(x - ct) + g(x + ct)$?

(**5**) The three dimensional wave equation for $\phi(x, y, z, t)$ is

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2},$$

where c is a constant. In a particular case $\phi = \phi(r, t)$ where (as usual) $r = \sqrt{(x^2+y^2+z^2)}$. Show successively that

$$\begin{aligned} (i) \quad \frac{\partial r}{\partial x} &= x r^{-1}, \\ (ii) \quad \frac{\partial \phi}{\partial x} &= x r^{-1}\frac{\partial \phi}{\partial r}, \\ (iii) \quad \frac{\partial^2 \phi}{\partial x^2} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{x^2}{r^2}\left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r}\frac{\partial \phi}{\partial r}\right). \end{aligned}$$

Deduce that $\phi = \phi(r, t)$ satisfies the *spherical wave equation*

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r}.$$

Find the PDE satisfied by $\psi = r\phi$; hence find the general solution of the spherical wave equation.

(**6**)The PDE

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} + \frac{1}{c^2 \tau}\frac{\partial y}{\partial t},$$

where τ is a positive constant, models waves on a string when there is friction. Find all (non-trivial) solutions of the PDE of the form $y(x, t) = X(x)T(t)$ given that $y(0, t) = y(L, t) = 0$. (Consider separately the cases (i) $2\pi c\tau > L$, (ii) $2\pi c\tau < L$ and $\exists$ *no* integer n with $2n\pi c\tau = L$, (iii) $2\pi c\tau \le L$ and $\exists$ n with $2n\pi c\tau = L$).