Conservation, Shocks and Characteristics in MHD

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With special thanx to ...!

Outline

1. Conservative Form
   - Basics of conservative form
   - Conservative MHD
   - Conservation laws
   - Dissipative MHD

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The standard ideal MHD equations ($\rho, \mathbf{v}, p, \mathbf{B}$)

Conservation of mass

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Momentum equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \nabla p - \rho \mathbf{g} - \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0,$$

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Alfvén's equation

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \qquad \nabla \cdot \mathbf{B} = 0.$$

Thermodynamics variables ($\rho, p \rightarrow e, s$)

$\rho, \mathbf{v}, p, \mathbf{B} \rightarrow \rho, \mathbf{v}, e, \mathbf{B}$

$$e \equiv \frac{1}{\gamma - 1}\frac{p}{\rho}(\approx C_v T), \qquad C_v \approx \frac{(1+Z)k}{(\gamma - 1)m_i}$$

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$\rho, \mathbf{v}, p, \mathbf{B} \rightarrow \rho, \mathbf{v}, s, \mathbf{B}$

$$s \equiv C_v \ln S + \mathrm{const}, \qquad S \equiv p\rho^{-\gamma}$$

+ CM and IE $\Rightarrow$

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = 0.$$

A crucial question

Do these equations provide us with *a complete model* for plasma dynamics?

Answer: a qualified **NO**, because

- What is actually the physical problem we want to solve?
- How is this problem mathematically translate in conditions to be imposed on the solutions of the PDEs?

Defining conservative form

Definition (Conservative form)

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Note:

When we say a generalized divergence, we refer to the inclusion of both time and space derivatives in the equation.

Motivation for conservative form

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1. When equations are written in conservative form, it becomes simple to obtain local and global conservation laws from them.
2. It also allows the jump conditions at shocks to be determined.
3. It makes available powerful numerical algorithms to solve the equations.

The MHD equations

- In their standard form, the ideal MHD equations are not all conservative.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p - \mathbf{j} \times \mathbf{B} &= -\rho \nabla \mathbf{\Phi}, \\ \mathbf{j} = \nabla \times \mathbf{B}, \qquad p &= (\gamma - 1)\, \rho e \end{aligned}$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1)\, e \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \qquad \mathbf{E} = -\mathbf{v} \times \mathbf{B}, \qquad \nabla \cdot \mathbf{B} = 0$$

The MHD equations

- Only the mass conservation and magnetic field equations are in conservative form.

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The MHD equations

- The remaining equations will require manipulation if they are to be expressed in the required conservative form.

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- In solar MHD this is normally a reasonable assumption. Whenever we consider some small region of the Sun, its internal gravitational attraction will be negligible compared to the gravitational attraction exerted by the rest of the Sun.
- This potential will however prevent us from finding a strictly conservative form for the momentum equation. This is of course because the external force allows momentum to be injected into the system, without taking account of the opposite change in momentum experienced by the external system.

Transformation to conservation form

Recall a few vector identities

$$\begin{aligned}
\nabla\cdot(\mathbf{ab}) &= \mathbf{a}\cdot\mathbf{b}+\mathbf{b}\nabla\cdot\mathbf{a},\\
\mathbf{a}\times(\nabla\times\mathbf{b}) &= (\nabla\mathbf{b})\cdot\mathbf{a}-\nabla\cdot(\mathbf{ab})-\mathbf{b}\nabla\cdot\mathbf{a},\\
\nabla\times(\mathbf{a}\times\mathbf{b}) &= \nabla\cdot(\mathbf{a}-\mathbf{ab}),\\
\nabla(\mathbf{a}\cdot\mathbf{b}) &= (\nabla\mathbf{a})\cdot\mathbf{b}+(\nabla\mathbf{b})\cdot\mathbf{a}.
\end{aligned}$$

The MHD equations

Rewrite of momentum equation

First two terms in ME

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial}{\partial t}(\rho\mathbf{v}) + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) + \rho\mathbf{v}\cdot\nabla\mathbf{v}$$
$$= \frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\mathbf{v}\mathbf{v})$$

Last term in ME

$$-\mathbf{j}\times\mathbf{B} = \mathbf{B}\times(\nabla\times\mathbf{B}) = \nabla(\frac{1}{2}B^2) - \nabla\cdot(\mathbf{B}\mathbf{B})$$

From Farady's law (for Alfvén's equation)

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The MHD equations

Conservation of energy

Internal energy cannot be brought into conservation form because it only contains part of energy $\Rightarrow$ need a conservation energy for the total energy.

$\mathbf{v}\cdot$ME $\rightarrow$

$$\rho\mathbf{v}\left(\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)+\mathbf{v}\cdot\nabla p-\mathbf{v}\cdot\mathbf{j}\times\mathbf{B}=-\rho\mathbf{v}\cdot\nabla\Phi$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right)-\frac{1}{2}v^2\frac{\partial\rho}{\partial t}+\frac{1}{2}\rho\mathbf{v}\cdot\nabla v^2+\mathbf{v}\cdot\nabla p-\mathbf{v}\cdot\mathbf{j}\times\mathbf{B}=-\rho\mathbf{v}\cdot\nabla\Phi$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right)+\nabla\cdot\left(\frac{1}{2}\rho v^2\mathbf{v}\right)+\mathbf{v}\cdot\nabla p-\mathbf{v}\cdot\mathbf{j}\times\mathbf{B}=-\rho\mathbf{v}\cdot\nabla\Phi,$$

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ρ IE $\rightarrow$

$$\rho \frac{\partial e}{\partial t} + \rho \mathbf{v} \cdot \nabla e + (\gamma - 1)\, \rho e \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial (\rho e)}{\partial t} - e \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla e + p \nabla \cdot \mathbf{v} = 0$$

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$\mathbf{B}\cdot$ AE $\rightarrow$

$$\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}-\mathbf{B}\cdot\nabla\times(\mathbf{v}\times\mathbf{B})=0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}B^2\right)+\nabla\cdot[\mathbf{B}\times(\mathbf{v}\times\mathbf{B})]-(\mathbf{v}\times\mathbf{B})\cdot\nabla\times\mathbf{B}=0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}B^2\right)+\nabla\cdot[\mathbf{B}\cdot\mathbf{B}\mathbf{v}-\mathbf{v}\cdot\mathbf{B}\mathbf{B}]+\mathbf{v}\cdot\mathbf{j}\times\mathbf{B}=0.$$

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- The momentum equation becomes:

$$\begin{aligned} \frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \left[\rho \mathbf{v}\mathbf{v} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{B}\mathbf{B} \right] &= -\rho \nabla \mathbf{\Phi}, \\ p &= (\gamma - 1)\, \rho e \end{aligned}$$

The conservative form of the MHD equations

By combining all fiddly vector algebra, we can now transform the MHD equations into vector form.

- The conservative form of the total energy equation becomes:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e + \frac{1}{2}B^2\right) + \nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho e + p + B^2\right)\mathbf{v} - \mathbf{v}\cdot\mathbf{BB}\right] = -\rho\mathbf{v}\cdot\nabla\mathbf{\Phi}$$

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- Finally the magnetic flux (Alfvén) equation becomes:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla\cdot(\mathbf{vB} - \mathbf{Bv}) = 0, \qquad \nabla\cdot\mathbf{B} = 0$$

The conservative form of the MHD equations

Note on entropy equation

For adiabatic processes of ideal gases neither S (entropy) or s (entropy per unit mass, or specific entropy) are conserved. However, introducing ρS (entropy per unit volume):

$$\frac{\partial}{\partial t}(\rho S) + \nabla \cdot (\rho s \mathbf{v}) = 0,$$

which is a conservation form.

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Definition (Stress tensor)

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Definition (Total energy density)

$$\mathcal{H} = \underbrace{\frac{1}{2}\rho v^2}_{\text{KEd}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{IEd}} + \underbrace{\frac{1}{2}B^2}_{\text{MEd}}$$

Physical meaning of conservative form

Based on the conservative form of the MHD equations, it is appropriate to define some additional quantities:

Definition (Energy flow)

$$\mathbf{U} = \left(\frac{1}{2}\rho v^2 + \rho e + p\right)\mathbf{v} + B^2\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\mathbf{B}$$

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Definition (No-name)

$$\mathbf{Y} = \mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}$$

Conservative evolution equations for ρ, π, $\mathcal{H}$, and **B**

Using these new variables, we can rewrite the ideal MHD equations, which become:

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$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \mathbf{U} = 0,$$ conservation of energy

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{Y} = 0,$$ conservation of magnetic flux

Divergence fixing

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$$p = (\gamma - 1)\left[\mathcal{H} - \frac{1}{2}\left(\frac{\pi^2}{\rho} + B^2\right)\right]$$

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presents a numerical challenge, as there is no numerical guarantee of a positive value of p being returned. The handling of negative pressure is an important feature of acceptable numerical schemes, in particular in the case of low β plasmas.

The stress tensor & energy flow

- If we look at the stress tensor

$$\mathbf{T} = \rho\mathbf{v}\mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathbf{I} - \mathbf{B}\mathbf{B}$$

we can see that it consists of $\rho\mathbf{v}\mathbf{v}$ the Reynolds stress tensor, $p\mathbf{I}$ the isotropic pressure and $\frac{1}{2}B^2\mathbf{I} - \mathbf{B}\mathbf{B}$ the magnetic component of the Maxwell stress tensor. The Reynolds stress represents a positive stress (pressure) of ρv^2 along $\mathbf{v}$, while the magnetic field provides a positive stress (pressure) perpendicular to $\mathbf{B}$ and a negative stress (magnetic tension) parallel to $\mathbf{B}$.

The stress tensor & energy flow

- The different terms of the total energy density $\mathcal{H}$ may be grouped into two parts

$$\mathcal{H} = \mathcal{K} + \mathcal{W}$$

where $\mathcal{K}$ is the kinetic energy density,

$$\mathcal{K} \equiv \frac{1}{2}\rho v^2,$$

and $\mathcal{W}$ is the potential energy density,

$$\mathcal{W} \equiv \rho e + \frac{1}{2}B^2 = \frac{p}{\gamma - 1} + \frac{1}{2}B^2$$

The stress tensor & energy flow

- The energy flow also consists of a hydrodynamic and magnetic part. The magnetic element of the flow

$$\mathbf{U}_{mag} \equiv B^2\mathbf{v} - \mathbf{v} \cdot \mathbf{BB}$$

may be transformed into the Poynting vector,

$$\mathbf{U}_{mag} = -(\mathbf{v} \times \mathbf{B}) \times \mathbf{B} = \mathbf{E} \times \mathbf{B} \equiv \mathbf{S}$$

which represents the flow of electromagnetic energy.

Global values

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Definition (Total magnetic flux)

$$\psi \equiv \int \mathbf{B} \cdot \tilde{\mathbf{n}} \mathrm{d}\tilde{\sigma}$$

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where $\int \mathrm{d}\tau$ is a volume integral and $\int \mathrm{d}\tilde{\sigma}$ is a surface integral over a cross-section of the plasma enclosed by some boundary curve $\oint \mathrm{d}\mathbf{l}$ which lies within the wall.

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$$\begin{aligned} \mathbf{F} = \dot{\mathbf{\Pi}} = \int \dot{\pi} \mathrm{d}\tau \quad &= \quad -\int \nabla \cdot \mathbf{T} \mathrm{d}\tau \\ \underbrace{=}_{\text{GT}} \quad & -\oint \left(p + \frac{1}{2} B^2 \right) \mathbf{n} \mathrm{d}\sigma \end{aligned}$$

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$$\underbrace{=}_{\text{GT}} \quad -\oint \left(p + \frac{1}{2} B^2 \right) \mathbf{n} \mathrm{d}\sigma = 0$$

Global conservation

We can find the derivatives of the global values thusly:

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$$\dot{H} = \int \dot{\mathcal{H}}\,\mathrm{d}\tau = -\nabla\cdot\mathbf{U}\,\mathrm{d}\tau \underbrace{=}_{\text{GT}} -\oint \mathbf{U}\cdot\mathbf{n}\,\mathrm{d}\sigma = 0$$

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We can find the derivatives of the global values thusly:

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$$\dot{\Psi} = \int \dot{\mathbf{B}} \cdot \tilde{\mathbf{n}} \mathrm{d}\tilde{\sigma} = \int \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot \tilde{\mathbf{n}} \mathrm{d}\tilde{\sigma} \underbrace{=}_{\text{ST}} \oint \mathbf{v} \times \mathbf{B} \cdot \mathrm{d}l = 0$$

Global conservation cont

- We see that the derivatives of all four of our global quantities, mass, momentum, energy and magnetic flux go to zero as a result of our boundary conditions $\mathbf{v} \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{B} = 0$, i.e. they are conserved and the system is closed. This is also true in the case of a plasma surrounded by a vacuum.

Global conservation cont

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- On the other hand, when modelling solar atmospheric plasmas we may want to incorporate overall changes in mass, momentum, energy and flux, in particular either through changes from below (the photosphere) or from above (the solar wind).

Kinematic expressions for motion of geometric elements

We take Lagrangian derivatives of line, surface and volume elements respectively (e.g. another advanced vector-algebra!):

- $$\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\mathbf{l}) = \mathrm{d}\mathbf{l} \cdot \nabla \mathbf{v}$$
- $$\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\boldsymbol{\sigma}) = -(\nabla \mathbf{v}) \cdot \mathrm{d}\boldsymbol{\sigma} + \nabla \cdot \mathbf{v}\mathrm{d}\boldsymbol{\sigma}$$
- $$\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\tau) = \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{v}\mathrm{d}\tau$$

Local (Lagrangian) conservation

Using these results, we can go on to take the Lagrangian derivatives of fluid elements in our global values:

- Motion of the *mass of fluid element*, $\mathrm{d}M \equiv \rho \mathrm{d}\tau$

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}M) &= \frac{\mathrm{D}}{\mathrm{D}t}(\rho \mathrm{d}\tau) = \frac{\mathrm{D}\rho}{\mathrm{D}t}(\rho \mathrm{d}\tau) + \rho \frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\tau) \\
&= -\rho \nabla \cdot \mathbf{v} \mathrm{d}\tau + \rho \nabla \cdot \mathbf{v} \mathrm{d}\tau = 0
\end{aligned}
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- Rate of change of *momentum of fluid element*, $\mathrm{d}\mathbf{\Pi} \equiv \boldsymbol{\pi} \mathrm{d}\tau$

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\mathbf{\Pi}) &= \frac{\mathrm{D}\pi}{\mathrm{D}t}\mathrm{d}\tau + \pi \frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\tau) \\ &= -\nabla \cdot \left[\rho \mathbf{v}\mathbf{v} + \left(+\frac{1}{2}B^2\right)\mathbf{I} - \mathbf{B}\mathbf{B} - \mathbf{v}\boldsymbol{\pi}\right] \mathrm{d}\tau \\ &= \left(-\nabla p + \mathbf{j} \times \mathbf{B}\right) \mathrm{d}\tau \neq 0\end{aligned}$$

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Using these results, we can go on to take the Lagrangian derivatives of fluid elements in our global values:

- Rate of change of *energy of fluid element*, $\mathrm{d}H \equiv \mathcal{H}\mathrm{d}\tau$

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}H) &= \frac{\mathrm{D}\mathcal{H}}{\mathrm{D}t}\mathrm{d}\tau + \mathcal{H}\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\tau) \\
&= -\nabla \cdot (\mathbf{U} - \mathbf{v}\mathcal{H})\,\mathrm{d}\tau \\
&= -\nabla \cdot \left[\left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + p + B^2\right)\mathbf{v} - \mathbf{v}\cdot\mathbf{BB}\right. \\
&- \left.\mathbf{v}\left(\frac{1}{2}\rho v^2 + \frac{p}{\gamma - 1} + \frac{1}{2}B^2\right)\right]\mathrm{d}\tau \\
&= -\nabla\left\{\left[\left(p + \frac{1}{2}B^2\right)\mathbf{I} - \mathbf{BB}\right]\cdot\mathbf{v}\right\}\mathrm{d}\tau \neq 0
\end{aligned}
$$

Local (Lagrangian) conservation

Taking the Lagrangian derivative of magnetic flux is important, but more complicated. We consider the motion of a surface element rather than a volume element, and our artificial variable **Y** is of no use to us, so, instead:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B} \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B}$$

so that

$$\frac{\mathrm{D}\mathbf{B}}{\mathrm{D}t} \equiv \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B} \nabla \cdot \mathbf{v}$$

Local (Lagrangian) conservation

- From above and the kinematic expression for surface element the *magnetic flux through a surface element* $d\psi \equiv \mathbf{B} \cdot d\boldsymbol{\sigma}$:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\mathrm{d}\psi) = \frac{\mathrm{D}}{\mathrm{D}t}(\mathbf{B} \cdot \mathrm{d}\boldsymbol{\sigma}) = \frac{\mathrm{D}\mathbf{B}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{\sigma} + \mathbf{B}\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\sigma})$$
$$= (\mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{B}\nabla \cdot \mathbf{v}) \cdot \mathrm{d}\boldsymbol{\sigma} + \mathbf{B} \cdot (-(\nabla \mathbf{v}) \cdot \mathrm{d}\boldsymbol{\sigma} + \nabla \cdot \mathbf{v}\mathrm{d}\boldsymbol{\sigma}) = 0$$

Hence the magnetic flux through a co-moving surface element is constant.

Local (Lagrangian) conservation

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$$= (\mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{B}\nabla\cdot\mathbf{v})\cdot\mathrm{d}\boldsymbol{\sigma} + \mathbf{B}\cdot(-(\nabla\mathbf{v})\cdot\mathrm{d}\boldsymbol{\sigma} + \nabla\cdot\mathbf{v}\mathrm{d}\boldsymbol{\sigma}) = 0$$

Hence the magnetic flux through a co-moving surface element is constant.
But this holds for *any* surface element, the flux through any surface bounded by a contour C moving with the fluid is conserved

$$\psi = \int_C \mathbf{B}\cdot\mathbf{n}\mathrm{d}\sigma = \text{const.}$$

Locally (Lagrangian) conserved values

Summary: From the previous slides, we conclude that:

1. The mass of a moving fluid element is constant.
2. The momentum of a moving fluid element changes through the force density acting on it.
3. The energy of a moving fluid element changes through the work done on it by the total pressure.
4. Non-trivially, the magnetic flux through a moving surface element is constant.

Frozen in field lines

Exciting conclusion of the section

We may consider shrinking the cross-sectional area of our surface element to become arbitrarily small; in this limit we are now talking about the dynamics of single magnetic field lines.

Frozen in field lines

Exciting conclusion of the section

Combining the Lagrangian forms of the mass and induction equations $\Rightarrow$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{1}{\rho}\left(\mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{B}\nabla\cdot\mathbf{v}\right) + \frac{\mathbf{B}}{\rho}\nabla\cdot\mathbf{v} = \frac{\mathbf{B}}{\rho}\cdot\nabla\mathbf{v}$$

and if we compare this to the kinematic expression for a line element

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\mathrm{d}\mathbf{l}\right) = \mathrm{d}\mathbf{l}\cdot\nabla\mathbf{v}$$

we see that a line element $\mathrm{d}\mathbf{l} \parallel \mathbf{B}$ moves in the same way as the quantity $\mathbf{B}/\rho$. *"The lines of force are thus 'frozen' in the body"* - Alfvén (1950).

Resistive MHD

- The resistive MHD equations are not at all conservative.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} + \nabla p - \mathbf{j} \times \mathbf{B} = -\rho \nabla \mathbf{\Phi},$$

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}, \qquad p = (\gamma - 1) \rho e$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) p + \gamma p \nabla \cdot \mathbf{v} = (1 - \gamma) \frac{\eta}{\mu_0^2} (\nabla \times \mathbf{B})^2$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0, \qquad \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}, \qquad \nabla \cdot \mathbf{B} = 0$$

Resistive MHD

- The mass conservation and magnetic solenoidal equations are in conservative form.

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- The remaining equations will require, again, manipulation if to be expressed in required (near)conservative form.

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$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}, \qquad p = (\gamma - 1) \rho e$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1) e \nabla \cdot \mathbf{v} = \frac{1}{\rho} \eta j^2$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu_0} \nabla \times (\eta \nabla \times \mathbf{B}), \qquad \nabla \cdot \mathbf{B} = 0$$

Resistive MHD

Energy equation

Kinetic energy contribution is same as in iMHD $\rightarrow$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \nabla \cdot \left(\frac{1}{2}\rho v^2 \mathbf{v}\right) + \mathbf{v} \cdot \nabla p - \mathbf{v} \cdot \mathbf{j} \times \mathbf{B} = -\rho \mathbf{v} \cdot \nabla \mathbf{\Phi}$$

Resistive MHD

Energy equation

Internal energy contribution replaced $\rightarrow$

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) + p \nabla \cdot \mathbf{v} = \eta j^2$$

Resistive MHD

Energy equation

Magnetic energy contribution replaced $\rightarrow$

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\nabla \cdot [\mathbf{B} \times (\mathbf{v} \times \mathbf{B})] - \mathbf{v} \cdot \mathbf{j} \times \mathbf{B}$$

$$= -\frac{1}{\mu_0}\mathbf{B} \cdot \nabla \times (\eta \mathbf{j})$$

$$= -\frac{1}{\mu_0}\nabla \cdot (\eta \mathbf{j} \times \mathbf{B}) - \eta j^2$$

Resistive MHD

- By combining all three contributions, we can now find the conservation form of the total energy in resistive MHD:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho e + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho v^2 + \rho e + p\right)\mathbf{v} + \frac{1}{\mu_0}\left(-\mathbf{v}\times\mathbf{B} + \eta\mathbf{j}\right)\times\mathbf{B}\right] = -\rho\mathbf{v}\cdot\nabla\mathbf{\Phi}$$

Note: Magnetic energy may be converted into internal energy, but the sum is constant.

Resistive MHD

- In contrast, the magnetic flux equation becomes essentially non-conservative in resistive MHD, due to the magnetic diffusivity coefficient η/μ_0

$$\frac{\partial \mathbf{B}}{\partial t}+\nabla\cdot(\mathbf{vB}-\mathbf{Bv}) = -\nabla\times\left(\frac{\eta}{\mu_0}\nabla\times\mathbf{B}\right) = \frac{\eta}{\mu_0}\nabla^2\mathbf{B}+\mathbf{j}\times\nabla\eta, \quad \nabla\cdot\mathbf{B} = 0$$

Resistive MHD

- Finally, the entropy conservation equations of iMHD change into equations that clearly exhibit non-conservation due to Ohmic dissipation

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = (\gamma - 1)\rho^{-\gamma}\eta j^2,$$

$$\frac{\partial(\rho S)}{\partial t} + \nabla \cdot (\rho S \mathbf{v}) = (\gamma - 1)\rho^{-\gamma+1}\eta j^2.$$

Outline

What is a shock?

- Sometimes we may want to work out what will happen to the MHD variables at a point where one or other of them becomes discontinuous, i.e. when an MHD shock forms.
- This is an important application of the conservative form of the MHD equations.
- A shock is an irreversible (entropy increasing) transition. Nevertheless, our conservation rules for mass, energy, momentum and flux should still apply across shocks.

Simple shocks

- In gas dynamics a shock transition is associated with supersonic flow upstream of the shock and subsonic flow downstream of the shock.
  Rankine-Hugoniot relations gives connection of HD variables.
  Role of Mach number $M \equiv v/c$, where c (sound speed) is characteristic.

Simple shocks

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  Rankine-Hugoniot relations gives connection of HD variables.
  Role of Mach number $M \equiv v/c$, where c (sound speed) is characteristic.
- In MHD the situation is rather richer as there are three characteristic speeds.

Simple shocks

- First consider perturbations in a 1D gas flow travelling at the local sound speed $c \equiv \sqrt{\gamma p/\rho}$. Their trajectories in $x - t$ plane (or characteristics) are straight lines with derivative $\frac{\mathrm{d}x}{\mathrm{d}t} = \pm c$.

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- However, if the sound speed changes suddenly (e.g. increased local pressure) $\Rightarrow$ slopes of characteristic decrease, and, one would anticipate that at some point, unless interfered with, the characteristics would cross.
- As a result, information from different points in space time accumulates, and thus gradients in our macroscopic variables build up.

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- However, if the sound speed changes suddenly (e.g. increased local pressure) $\Rightarrow$ slopes of characteristic decrease, and, one would anticipate that at some point, unless interfered with, the characteristics would cross.
- As a result, information from different points in space time accumulates, and thus gradients in our macroscopic variables build up.
- At some point, non-linear effects will be balanced by dissipative effects in a narrow layer; in this steady-state, a shock wave has been generated.

Shock relations

- If we assume that the thickness, δ, of the shock is negligible then the steady state consists of two regions of different sound speed, separated by a shock front.

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Shock relations

- If we assume that the thickness, δ, of the shock is negligible then the steady state consists of two regions of different sound speed, separated by a shock front.
- Without actually specifying the type of dissipation in the shock front, we may still derive *shock relations* which relate the values of our macroscopic variable son either side of the front.
- We assume that although the ideal model breaks down in an infinitesimally thin layer in the shock, it still holds on either side of this layer.

Jumps

Across the negligibly thin ($\delta \to 0$) non-ideal layer MHD variables *jump*. The magnitude of the jumps may be determined by conservation of mass, momentum, energy and magnetic flux across the shock layer.

Definition

Jump in a variable f

$$[[f]] = f_1 - f_2$$

Jumps

- Integrate the spatial derivative across the shock, keeping only contributions normal (**n**, where undisturbed fluid is *ahead*) to the shock front as these gradients are infinitely large, then

$$\lim_{\delta\to 0}\int_1^2 \nabla f \mathrm{d}l = -\lim_{\delta\to 0}\mathbf{n}\int_1^2 \frac{\partial f}{\partial l}\mathrm{d}l = \mathbf{n}\,[[f]]$$

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- Considering the full derivative in the shock frame moving with the normal speed u,

$$\left(\frac{\mathrm{D}f}{\mathrm{D}t}\right)_{shock} = \frac{\partial f}{\partial t} - u\frac{\partial f}{\partial l} \ll \infty$$

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Though indiv. time/space derivatives must both be infinitely large, their balance should not be. Hence we need

$$\lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial t}\mathrm{d}l = u\lim_{\delta\to 0}\int_1^2 \frac{\partial f}{\partial l}\mathrm{d}l = -u\,[[f]]$$

Shock conservation equations

In conclusion: shock relations are obtained therefore by substitution of derivatives into the conservation equations

$$\frac{\partial f}{\partial t} \rightarrow -u\,[[f]]\,, \qquad\qquad \nabla f \rightarrow \mathbf{n}\,[[f]]$$

Shock conservation equations

The resulting full iMHD jump conditions are

$$-u\,[[\rho]] + \mathbf{n}\cdot[[\rho\mathbf{v}]] = 0$$

$$-u\,[[\rho\mathbf{v}]] + \mathbf{n}\cdot\left[\left[\rho\mathbf{v}\mathbf{v} + \left(p + \frac{1}{2}B^2\right)\mathbf{I} - \mathbf{B}\mathbf{B}\right]\right] = 0$$

$$-u\left[\left[\frac{1}{2}\rho v^2 + \frac{p}{\gamma-1} + \frac{1}{2}B^2\right]\right] + \mathbf{n}\cdot\left[\left[\left(\frac{1}{2}\rho v^2 + \frac{\gamma}{\gamma-1}p + B^2\right)\mathbf{v} - \mathbf{v}\cdot\mathbf{B}\mathbf{B}\right]\right] = 0$$

$$-u\,[[\mathbf{B}]] + \mathbf{n}\cdot[[\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}]] = 0, \qquad \mathbf{n}\cdot[[\mathbf{B}]] = 0$$

Entropy

Change of entropy at a shock

The second law of thermodynamics demands that entropy must not decrease after the passage of a shock, and therefore we must add a further condition to our jump conditions above.

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This condition is all that remains from the dissipative processes in the infinitesimal limit of the boundary layer.

Conditions in the shock frame

Transforming to the shock frame (steady shocks), the fluid velocity becomes $\mathbf{v}' \equiv \mathbf{v} - u\mathbf{n}$ and the jump conditions may be rewritten w.r.t. this frame:

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$$[[\rho v_n']] = 0 \qquad mass$$

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$$[[\rho v_n']] = 0 \qquad \textit{mass}$$

$$\left[\left[\rho v_n' + p + \frac{1}{2}B_t^2\right]\right] = 0 \qquad \textit{normal momentum}$$

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$$\left[\left[\rho v_n' + p + \frac{1}{2}B_t^2\right]\right] = 0 \qquad \textit{normal momentum}$$

$$\rho v_n' \left[\left[v_t'\right]\right] = B_n \left[\left[\mathbf{B}_t\right]\right] \qquad \textit{tangential momentum}$$

Conditions in the shock frame

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$$\rho v'_n \left[\left[\frac{1}{2}\left({v'_n}^2 + {v'_t}^2\right) + \frac{1}{\rho}\left(\frac{\gamma}{\gamma - 1}p + B_t^2\right)\right]\right] = B_n \left[\left[\mathbf{v}'_t \cdot \mathbf{B}_t\right]\right] \quad \textit{energy}$$

Conditions in the shock frame

Transforming to the shock frame (steady shocks), the fluid velocity becomes $\mathbf{v}' \equiv \mathbf{v} - u\mathbf{n}$ and the jump conditions may be rewritten w.r.t. this frame:

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6 algebraic eqs for 6 jumps $[[\rho]], [[v_n]], [[\mathbf{v}_t]], [[p]], [[B_n]], [[\mathbf{B}_t]]$.

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Application: Jump conditions (CFs) for moving plasma-plasma interface, where $\nexists$ no flow across the discontinuity ($v_n' = 0$):

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coupled with the condition of increasing entropy.

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These jump conditions imply two distinct possibilities:

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Definition

Contact discontinuity: if $B_n \neq 0$ the field intersects the surface and our variables are alternatively:
-jumping:$[[\rho]] \neq 0$
-continuous:
$v'_n = 0, \quad [[\mathbf{v}'_t]] = 0, \quad [[p]] = 0, \quad [[B_n]] = 0, \quad [[\mathbf{B}_t]] = 0$

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Definition

Tangential discontinuity: if $B_n = 0$ the field is parallel to the surface and our variables are alternatively:
-jumping:$[[\rho]] \neq 0, \quad [[\mathbf{v}'_t]] \neq 0, \quad [[p]] \neq 0, \quad [[\mathbf{B}_t]] \neq 0$
-continuous: $v'_n = 0, \quad [[\mathbf{v}'_t]] = 0, \quad [[B_n]] = 0, \quad \left[\left[p + \frac{1}{2}B_t^2\right]\right]$

Relevance to real plasmas

- Based on these two types of discontinuity, we can distinguish two distinct magnetic configurations with an interface.

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- In a star (e.g. the sun) we will often find that magnetic fields originate inside the star but intersect the stellar (solar) photosphere. These jumps may be considered as contact discontinuities, and only density is discontinuous.
- In laboratory plasmas aimed at fusion, the discontinuity of interest may be the containment of a high density plasma by a lower density one, to thermally isolate it from an outer wall. These discontinuities are typically tangential, and permit much more freedom in the choice of variable values.

Outline

Sound waves

Take iMHD equations with $\mathbf{B} = 0$
Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

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Energy equation

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0,$$

Sound waves

Linearize about a time-independent $\partial/\partial t = 0$ infinite and homogeneous ($\nabla = 0$) background characterized by arbitrary, but constant $\rho_0, \mathbf{v}_0$, and p_0

$$\rho(\mathbf{r}, t) = \rho_0 + \rho_1(\mathbf{r}, t) \qquad (\text{where } \; |\rho_1| \ll \rho_0 = \text{const}),$$

$$p(\mathbf{r}, t) = p_0 + p_1(\mathbf{r}, t) \qquad (\text{where } \; |p_1| \ll p_0 = \text{const}),$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_0 + \mathbf{v}_1(\mathbf{r}, t) \qquad (\text{where } \; |\mathbf{v}_1| \ll 1),$$

Sound waves

Linearized iHD equations
Conservation of mass

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right) \rho_1 + \rho_0 \nabla \cdot \mathbf{v}_1 = 0,$$

Momentum equation

$$\rho_0 \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right) \mathbf{v}_1 + \nabla p_1 = 0,$$

Energy equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right) p_1 + \gamma p_0 \nabla \cdot \mathbf{v}_1 = 0.$$

Note: Conservation of mass decouples.

Sound waves

Wave equation for sound wave propagation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\right)^2 \mathbf{v}_1 - c^2 \nabla\nabla \cdot \mathbf{v}_1 = 0,$$

where $c \equiv \sqrt{\gamma p_0/\rho_0}$. WE has constant coefficients $(\mathbf{v}_0, c) \Rightarrow$ GS superposition of plane waves:

$$\mathbf{v}_1 = \sum_k \hat{\mathbf{v}}_\mathbf{k} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad \Rightarrow \qquad \nabla \to i\mathbf{k}, \qquad \partial/\partial t \to -i\omega.$$

This transforms WE PDE into algebraic EVP

$$[(\omega - \mathbf{k} \cdot \mathbf{v}_0)^2 \mathbf{I} - c^2 \mathbf{k}\mathbf{k}] \cdot \hat{\mathbf{v}}_\mathbf{k} = 0.$$

Sound waves

Solution to EVP for sound wave propagation: Since $\nexists$ preferred direction, we restrict the waves to propagate in the z-direction only, $\mathbf{k} = k\mathbf{e}_z$, $\Rightarrow$ EVP reduces to

$$\begin{aligned} \omega^2 \hat{v}_x &= 0, \\ \omega^2 \hat{v}_y &= 0, \\ (\omega^2 - k^2 c^2)\hat{v}_z &= 0. \end{aligned}$$

Solutions:

- $\omega + \pm kc, \qquad \hat{v}_x = \hat{v}_y = 0, \qquad \hat{v}_z$ arbitrary

Compressible waves ($\nabla \cdot \neq 0$) longitudinal ($\mathbf{v}_1 \parallel \mathbf{k}$) propagation.

- $\omega^2 = 0, \qquad \hat{v}_x = \hat{v}_y$ arbitrary, $\hat{v}_z = 0$

Time-independent incompressible transverse ($\mathbf{v}_1 \perp \mathbf{k}$) translation; no interesting physics.

Sound waves

Note: Let's count!

• 1^{st} order linear system: $\exists$ 5 degrees of freedom rep by the five primitive variable

• 2^{nd} order system: *appears* 6 degrees of freedom since there are three components of $\mathbf{v}_1$ and EV is squared

However, 2^{nd} order system only has 4 degrees of freedom (see above), since quadratic dependence on ω *does not* double the direction of translation $\Rightarrow$ we lost a degree of freedom.

• Solution: spurious root of doubling $\omega = 0$ came when applying $\mathrm{D}/\mathrm{D}t$ in terms of $\mathbf{v}_1$ only and CM (eq. for ρ_1) was dropped. Inserting $\mathbf{v}_1 = 0$ into orig PDEs, we recover this lost mode called entropy wave rep density perturbations:

$$\omega\hat{\rho} = 0 \Rightarrow \omega = 0, \quad \hat{\rho} \text{ arbitrary, but } \hat{\mathbf{v}} = 0 \text{ and } \hat{p} = 0.$$

MHD waves

- We already had two sets of variables: $\{\rho, \mathbf{v}, p, \mathbf{B}\}$ & $\{\rho, \mathbf{v}, e, \mathbf{B}\}$. Since conservation laws were based on the latter, we exploit this set of PDEs.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + (\gamma - 1)\nabla(\rho e) + (\nabla \mathbf{B}) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{B} = 0$$

$$\frac{\partial e}{\partial t} + \mathbf{v} \cdot \nabla e + (\gamma - 1)\, e \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} + \mathbf{B} \nabla \cdot \mathbf{v} - \mathbf{B} \cdot \nabla \mathbf{v} = 0, \qquad \nabla \cdot \mathbf{B} = 0.$$

MHD waves

Linearize about a time-independent $\partial/\partial t = 0$ infinite and homogeneous ($\nabla = 0$) background characterized by arbitrary, but constant $\rho_0, \mathbf{v}_0 = 0$ (i.e. *plasma at rest*), $e_0 \equiv p_0/[(\gamma - 1)\rho_0]$, and $\mathbf{B}_0$

$$\rho(\mathbf{r}, t) = \rho_0 + \rho_1(\mathbf{r}, t) \qquad (\text{where } |\rho_1| \ll \rho_0 = \text{const}),$$

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}_1(\mathbf{r}, t) \qquad (\text{where } |\mathbf{v}_1| \ll 1),$$

$$e(\mathbf{r}, t) = e_0 + e_1(\mathbf{r}, t) \qquad (\text{where } |e_1| \ll e_0 = \text{const}),$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r}, t) \qquad (\text{where } |\mathbf{B}_1| \ll \mathbf{B}_0 = \text{const}).$$

MHD waves

Linearized iMHD equations

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + (\gamma - 1)(\rho_0 \nabla e_1 + e_0 \nabla \rho_1) + (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla \mathbf{B}_1 = 0$$

$$\frac{\partial e_1}{\partial t} + (\gamma - 1) e_0 \nabla \cdot \mathbf{v}_1 = 0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0 \nabla \cdot \mathbf{v}_1 - \mathbf{B}_0 \cdot \nabla \mathbf{v}_1 = 0, \qquad \nabla \cdot \mathbf{B}_1 = 0.$$

MHD waves

Note

- Two characteristic speeds:

$$c \equiv \sqrt{\frac{\gamma p_0}{\rho_0}}, \qquad \mathbf{b} \equiv \frac{\mathbf{B}_0}{\sqrt{\rho_0}}$$

- Dimensionless variables:

$$\tilde{\rho} \equiv \frac{\rho_1}{\gamma \rho_0}, \qquad \tilde{\mathbf{v}} \equiv \frac{\mathbf{v}_1}{c}, \qquad \tilde{e} \equiv \frac{e_1}{\gamma e_0}, \qquad \tilde{\mathbf{B}} \equiv \frac{\mathbf{B}_1}{c\sqrt{\rho_0}}$$

such that the linearized MHD PDEs only involve the coefficients c and $\mathbf{b}$ (and γ).

MHD waves

Linearized dimensionless iMHD equations

$$\gamma \frac{\partial \tilde{\rho}}{\partial t} + c\nabla \cdot \tilde{\mathbf{v}} = 0,$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + c\nabla \tilde{e} + c\nabla \tilde{\rho} + (\nabla \tilde{\mathbf{B}}) \cdot \mathbf{b} - \mathbf{b} \cdot \nabla \tilde{\mathbf{B}} = 0,$$

$$\frac{\gamma}{\gamma - 1} \frac{\partial \tilde{e}}{\partial t} + c\nabla \cdot \tilde{\mathbf{v}} = 0,$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} + \mathbf{b}\nabla \cdot \tilde{\mathbf{v}} - \mathbf{b} \cdot \nabla \tilde{\mathbf{v}} = 0, \qquad \nabla \cdot \tilde{\mathbf{B}} = 0.$$

MHD waves

Let us consider plane wave solutions

$$\tilde{f}(\mathbf{r}, t) = \hat{f} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

$\Rightarrow$ algebraic system of EVP:

$$\begin{aligned}
c\mathbf{k}\cdot\hat{\mathbf{v}} &= \gamma\omega\hat{\rho}, \\
\mathbf{k}c\hat{\rho} + \mathbf{k}c\hat{e} + (\mathbf{k}\mathbf{b}\cdot - \mathbf{k}\cdot\mathbf{b})\hat{\mathbf{B}} &= \omega\hat{\mathbf{v}}, \\
c\mathbf{k}\cdot\hat{\mathbf{v}} &= \frac{\gamma}{\gamma-1}\omega\hat{e}, \\
(\mathbf{b}\mathbf{k}\cdot - \mathbf{b}\cdot\mathbf{k})\hat{\mathbf{v}} &= \omega\hat{\mathbf{v}} \qquad \mathbf{k}\cdot\hat{\mathbf{B}} = 0.
\end{aligned}$$

MHD waves

Notes

- System solvable if det[8×8 matrix] $= 0$
- Medium anisotropic since the presence of $\mathbf{B}_0$ (or $\mathbf{b}$)
- In general $\mathbf{k} \nparallel \mathbf{b}$
- No loss of generality if $\mathbf{b} \parallel \mathbf{e}_z$ & $\mathbf{k} \in [x, z]$, i.e.

$$\mathbf{b} = (0, 0, b), \qquad \mathbf{k} = (k_\perp, 0, k_\parallel)$$

MHD waves

$$\begin{pmatrix} 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ k_\perp c & 0 & 0 & 0 & k_\perp c & -k_\parallel b & 0 & k_\perp b \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_\parallel b & 0 \\ k_\parallel c & 0 & 0 & 0 & k_\parallel c & 0 & 0 & 0 \\ 0 & k_\perp c & 0 & k_\parallel c & 0 & 0 & 0 & 0 \\ 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_\parallel b & 0 & 0 & 0 & 0 & 0 \\ 0 & k_\perp b & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \\ \hat{e} \\ \hat{B}_x \\ \hat{B}_y \\ \hat{B}_z \end{pmatrix} = \omega \begin{pmatrix} \gamma\hat{\rho} \\ \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \\ \frac{\gamma}{\gamma-1}\hat{e} \\ \hat{B}_x \\ \hat{B}_y \\ \hat{B}_z \end{pmatrix}$$

MHD waves

Important note

- Observe the symmetry of operator describing the linearized system of iMHD
- Symmetry is related to the fact that nonlinear iMHD eqs are symmetric hyperbolic PDEs
- This symmetry properties were behind the preference for the set of $\{\rho, \mathbf{v}, e, \mathbf{B}\}$ representation
- This formalism has a natural extension (e.g., dissipation, etc) and is frequently exploited by large-scale numerical computations

MHD waves

Entropy wave

- Transfer set of variables $\{\hat{\rho}, \hat{\mathbf{v}}, \hat{e}, \hat{\mathbf{B}}\} \to \{\hat{S}, \hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{B}}\}$
- Corresponding EVP

$$\begin{aligned} -\omega\hat{S} &= 0, \\ -\omega\hat{\mathbf{v}} &+ \mathbf{k}c\hat{p} + (\mathbf{kb}\cdot - \mathbf{k}\cdot\mathbf{b})\hat{\mathbf{B}} = 0, \\ -\omega\hat{p} &+ c\mathbf{k}\cdot\hat{\mathbf{v}} = 0, \\ -\omega\hat{\mathbf{B}} &+ (\mathbf{bk}\cdot - \mathbf{b}\cdot\mathbf{k})\hat{\mathbf{v}} = 0, \qquad \mathbf{k}\cdot\hat{\mathbf{B}} = 0. \end{aligned}$$

- Genuine (also called marginal) entropy mode

$$\omega = 0, \qquad \hat{p} = \hat{e} + \hat{\rho} = 0, \qquad \hat{S} = \gamma\hat{e} = -\gamma\hat{\rho} \neq 0.$$

MHD waves

Magnetic field constraint

• Reminder: $\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{k} \cdot \hat{\mathbf{B}} = 0$
• EVP returns one spurious EV, $\omega = 0$, distinct from the root for entropy wave, suggesting corresponding eigenvectors satisfying $\mathbf{k} \cdot \hat{\mathbf{B}} \neq 0$ (see e.g. operating projector **k** onto AE) $\Rightarrow$ in numerical MHD due to truncation errors this spurious root would not be distinguishable from a genuine transition from stability to instability!
• Solution:
(i) Eliminate e.g. $\hat{B}_z$, i.e. $8 \times 8 \rightarrow 7 \times 7$
(ii) Wave vector projection

$$\hat{v}_{1,2} \equiv [(\mathbf{k}/k) \times \hat{\mathbf{v}}]_{x,y}, \qquad \hat{v}_z \equiv (\mathbf{k}/k) \cdot \hat{\mathbf{v}}, \qquad \hat{B}_{1,2} \equiv [(\mathbf{k}/k) \times \hat{\mathbf{B}}]_{x,y}.$$

MHD waves

This will result in an EVP of 7×7:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_{\|} b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -k_{\|} b \\ 0 & 0 & 0 & 0 & kc & -k_{\perp} b & 0 \\ 0 & 0 & 0 & kc & 0 & 0 & 0 \\ 0 & -k_{\|} b & 0 & -k_{\perp} b & 0 & 0 & 0 \\ 0 & 0 & -k_{\|} b & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{S} \\ \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \\ \hat{p} \\ \hat{B}_1 \\ \hat{B}_2 \end{pmatrix} = \omega \begin{pmatrix} \hat{S} \\ \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \\ \hat{p} \\ \hat{B}_1 \\ \hat{B}_2 \end{pmatrix}$$

Discarding entropy wave (i.e. dropping first raw and first column) $\Rightarrow$ 6×6 matrix.

MHD waves

Numerical 8-wave scheme with $\nabla \cdot \mathbf{B}$ wave

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \pi = 0,$$

$$\frac{\partial \pi}{\partial t} + \nabla \cdot \left[\pi \mathbf{v} + \left(p + \frac{1}{2} B^2 \right) \mathbf{I} - \mathbf{BB} \right] = -\mathbf{B} \nabla \cdot \mathbf{B},$$

$$\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \left[\mathcal{H} + \left(p + \frac{1}{2} B^2 \right) \mathbf{v} - \mathbf{v} \cdot \mathbf{BB} \right] = -\mathbf{v} \cdot \mathbf{B} \nabla \cdot \mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{vB} - \mathbf{Bv}) = -\mathbf{v} \nabla \cdot \mathbf{B}, \qquad \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) + \nabla \cdot (\mathbf{v} \nabla \cdot \mathbf{B}).$$

- Artificial extension of MHD system with $\nabla \cdot \mathbf{B}$ wave; $\nabla \cdot \mathbf{B}$ is convected with fluid (so, if it was small at BCs, it remains small).

MHD waves: Reduction to velocity representation- three waves

- More powerful, as it gets rid off entropy wave + absorbs the constraint $\mathbf{k}\cdot\hat{\mathbf{B}}$
- MHD wave equation in homogeneous medium

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - \left[(\mathbf{b}\cdot\nabla)^2\mathbf{I} + (b^2+c^2)\nabla\nabla - \mathbf{b}\cdot\nabla(\nabla\mathbf{b}+\mathbf{b}\nabla)\right]\cdot\mathbf{v}_1 = 0.$$

- The EV equation for plane waves:

$$\left\{\left[\omega^2 - (\mathbf{k}\cdot\mathbf{b})^2\right]\mathbf{I} - (b^2+c^2)\mathbf{k}\mathbf{k} + \mathbf{k}\cdot\mathbf{b}(\mathbf{k}\mathbf{b}+\mathbf{b}\mathbf{k})\right\}\cdot\hat{\mathbf{v}} = 0$$

MHD waves: Reduction to velocity representation- three waves

In components...

$$\begin{pmatrix} -k_\perp^2(b^2+c^2)-k_\parallel^2 b^2 & 0 & -k_\perp k_\parallel c^2 \\ 0 & -k_\parallel^2 b^2 & 0 \\ -k_\perp k_\parallel c^2 & 0 & -k_\parallel^2 b^2 \end{pmatrix} \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = -\omega^2 \begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix}$$

• 3 symmetric matrix with quadratic EV ω^2, corresponding to original 6×6 rep of ω.
• The reduction in terms of velocity alone may be generalized to inhomogeneous plasma $\rightarrow$ force operator formalism.
• With description in terms of velocity we lost the marginal entropy mode ($\omega = 0$). For sake of completeness we include in det$[3 \times 3]$ leading to Dispersion Equation for MHD waves $\Rightarrow$

MHD waves: Reduction to velocity representation- three waves

- Dispersion Equation

$$\det = \omega(\omega^2 - k_{\parallel}^2 b^2)\left[\omega^4 - k^2(b^2 + c^2)\omega^2 + k_{\parallel}^2 k^2 b^2 c^2\right] = 0,$$

with solutions $\omega = \omega_i(\mathbf{k})$ $(i = 1, \ldots, 7)$.

- DE admits four kind of solutions.

MHD waves: Reduction to velocity representation- three waves

- Dispersion Equation

$$\det = \omega(\omega^2 - k_{\parallel}^2 b^2)\left[\omega^4 - k^2(b^2 + c^2)\omega^2 + k_{\parallel}^2 k^2 b^2 c^2\right] = 0,$$

with solutions $\omega = \omega_i(\mathbf{k})$ $(i = 1, \ldots, 7)$.
(i) Entropy waves

$$\omega = \omega_E = 0, \qquad \hat{\mathbf{v}} = \hat{\mathbf{B}} = 0, \qquad \hat{p} = 0, \qquad \text{but} \qquad \hat{S} \neq 0.$$

MHD waves: Reduction to velocity representation- three waves

- Dispersion Equation

$$\det = \omega(\omega^2 - k_\parallel^2 b^2)\left[\omega^4 - k^2(b^2+c^2)\omega^2 + k_\parallel^2 k^2 b^2 c^2\right] = 0,$$

with solutions $\omega = \omega_i(\mathbf{k})$ $(i = 1, \dots, 7)$.
(ii) Alfvén waves

$$\omega = \pm\omega_A, \qquad \omega_A \equiv \mathbf{k}\cdot\mathbf{b} = k_\parallel \cos\vartheta.$$

$$\hat{B}_y = -\hat{v}_y \neq 0, \qquad \hat{v}_x = \hat{v}_z = \hat{B}_x = \hat{B}_z = \hat{S} = \hat{p} = 0.$$

MHD waves: Reduction to velocity representation- three waves

- Dispersion Equation

$$\det = \omega(\omega^2 - k_{\parallel}^2 b^2)\left[\omega^4 - k^2(b^2 + c^2)\omega^2 + k_{\parallel}^2 k^2 b^2 c^2\right] = 0,$$

with solutions $\omega = \omega_i(\mathbf{k})$ $(i = 1, \ldots, 7)$.
(iii) Fast and slow MA waves

$$\omega = \pm\omega_{s,f} \equiv \pm k\left(\frac{1}{2}(b^2 + c^2) \pm \frac{1}{2}\sqrt{(b^2 + c^2)^2 - 4(k_{\parallel}^2/k^2)b^2c^2}\right)^{1/2}$$

Characteristics: revision

- The method of characteristics

Linear advection equation

$$\frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} = 0$$

which for u =const (advection velocity) has the solution

$$\Psi = f(x - ut), \qquad \text{where} \quad f = \Psi_0 \equiv \Psi(x, t = 0),$$

ie. initial data Ψ_0 simply propagate along the set of parallel straight lines $dx/dt = u$, which are the characteristics.

- In (M)HD Ψ is a vector, u is a matrix.

Characteristics: revision

- E.g. in 1-D linear HD

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 0 & 1/\rho_0 \\ \gamma p_0 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ p_1 \end{pmatrix} = 0$$

Solution represents sound waves with frequencies $\omega = \pm kc$. Exploiting real notation, GS may be written (e.g. for v_1 or p_1)

$$v_1(x,t) = \sum_k \left[\alpha_k \sin k(x \pm ct) + \beta_k \cos k(x \pm ct)\right]$$

Here α_k, β_k follow from Fourier decomposition of the initial data,

$$v_1(x,0) = \sum_k \left[\alpha_k \sin kx + \beta_k \cos kx\right]$$

demonstrating that the initial data do, in fact, propagate along two sets of straight-line characteristics $dx/dt = \pm c$.

Characteristics: revision

Linear advection equation

$$\frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} = 0$$

If $u \neq$const, the characteristics become solutions of the ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u\left(x, t\right)$$

Along these curves, the solution $\Psi(x, t)$ is constant,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} \equiv \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = 0.$$

Characteristics: revision

Most importantly for (M)HD/C(M)FD the MoC can be generalized to nonlin PDEs. E.g. in quasi-linear advection eq, when u is also fnc of Ψ. In particular, in the case $u = \Psi$ this leads to Burger's equation:

$$\frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi}{\partial x} = \nu \frac{\partial^2 \Psi}{\partial x^2}$$

Here viscosity balances nonlinearity. When neglecting the small dissipative term, the characteristics are the solution of ODE

$$\mathrm{d}x/\mathrm{d}t = \Psi\left(x(t), t\right).$$

Characteristics have different slopes $\Rightarrow$ $t >> 1$ they cross $\Rightarrow$ leading to shock development. Recall: MHD PDEs are hyperbolic $\Rightarrow$ they possess a complete set of real characteristics related to the eigenvalues of the linearized

Characteristics in iMHD

• The iMHD eqs are PDEs w.r.t. the independent variables $\mathbf{r}, t$
$\Rightarrow$ characteristics will be 3-D manifolds

$$\xi\left(\mathbf{r}, t\right) = \xi_0$$

in 4-D space-time.
• Assume that boundary data for $\rho(\mathbf{r}, t), \mathbf{v}(\mathbf{r}, t), e(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$ are given on the manifold $\xi\left(\mathbf{r}, t\right) = \xi_0$.
• Let us consider ξ as a coordinate and introduce η, ζ, and τ boundary fitted coordinates that parameterize the boundary manifold $\xi\left(\mathbf{r}, t\right) = \xi_0$. $\Rightarrow$ the boundary data may be written as

$$\rho\left(\xi_0, \eta, \zeta, \tau\right) = \rho_0\left(\eta.\zeta, \tau\right), \qquad \text{etc.}$$

Since $\rho_0(\eta, \zeta, \tau)$ is a know fnc, the derivatives $(\partial\rho_0/\partial\eta)$, etc. may also be considered known. And similarly for $\mathbf{v}_0, e_0, \ldots$.

Characteristics in iMHD

- Mission: to find out under which conditions the solutions $\rho(\xi,\eta,\zeta,\tau), \mathbf{v}(\xi,\eta,\zeta,\tau), e(\xi,\eta,\zeta,\tau), \mathbf{B}(\xi,\eta,\zeta,\tau)$ may be obtained away from the boundary $\xi = \xi_0$. To that end:

$$\begin{aligned}\rho(\xi,\eta,\zeta,\tau) &= \rho_0(\eta_0,\zeta_0,\tau_0) + (\xi-\xi_0)\left(\frac{\partial\rho}{\partial\xi}\right)_0 \\ &+ (\eta-\eta_0)\left(\frac{\partial\rho}{\partial\eta}\right)_0 + (\zeta-\zeta_0)\left(\frac{\partial\rho}{\partial\zeta}\right)_0 \\ &+ (\tau-\tau_0)\left(\frac{\partial\rho}{\partial\tau}\right)_0 + \dots\end{aligned}$$

Mission completed if $(\partial\rho/\partial\xi)_0$, $(\partial\mathbf{v}/\partial\xi)_0$, $(\partial e/\partial\xi)_0$, and $(\partial\mathbf{B}/\partial\xi)_0$ can be constructed, since the other 1^{st}-order deriv.s are found from boundary data, whereas higher-order deriv.s may be obtained by subsequent differentiation of orig PDE.

Characteristics in iMHD

- Denote the unknown derivatives w.r.t. ξ:

$$\rho' \equiv \frac{\partial \rho}{\partial \xi}, \qquad \mathbf{v}' \equiv \frac{\partial \mathbf{v}}{\partial \xi}, \qquad e' \equiv \frac{\partial e}{\partial \xi}, \qquad \mathbf{B}' \equiv \frac{\partial \mathbf{B}}{\partial \xi}$$

- Derivatives of MHD eqs may be written:

$$\begin{aligned} \nabla\rho &= \nabla\xi\rho' + \nabla\eta\frac{\partial\rho}{\partial\eta} + \nabla\zeta\frac{\partial\rho}{\partial\zeta} + \nabla\tau\frac{\partial\rho}{\partial\tau} \\ \frac{\mathrm{D}\rho}{\mathrm{D}t} &= (\xi_t + \mathbf{v}\cdot\nabla\xi)\,\rho' + (\eta_t + \mathbf{v}\cdot\nabla\eta)\,\frac{\partial\rho}{\partial\eta} \\ &\quad + (\zeta_t + \mathbf{v}\cdot\nabla\zeta)\,\frac{\partial\rho}{\partial\zeta} + (\tau_t + \mathbf{v}\cdot\nabla\tau)\,\frac{\partial\rho}{\partial\tau} \end{aligned}$$

Characteristics in iMHD

• W.r.t. the primed variables the coord. transform can be summarized:

$$\begin{aligned} \nabla f &\to \mathbf{n} f' + \dots, & \mathbf{n} &\equiv \nabla\xi, \\ \frac{\mathrm{D}f}{\mathrm{D}t} &\to -uf' + \dots, & -u &\equiv \xi_t + \mathbf{v}\cdot\nabla\xi \end{aligned}$$

• Here **n** is the normal to the space-part of the characteristics, and u is the characteristic speed (i.e. normal velocity of characteristic ξ measured w.r.t. fluid velocity **v**).

• Inserting these to iMHD PDEs $\Rightarrow$

Characteristics in iMHD

$$\begin{aligned} -u\rho' + \rho\mathbf{n}\cdot\mathbf{v}' &= \ldots \\ -\rho u\mathbf{v}' + (\gamma-1)\,\mathbf{n}\left(e\rho' + \rho e'\right) + (\mathbf{nB}\cdot - \mathbf{n}\cdot\mathbf{B})\,\mathbf{B}' &= \ldots \\ -ue' + (\gamma-1)\,e\mathbf{n}\cdot\mathbf{v}' &= \ldots \\ -u\mathbf{B}' + (\mathbf{Bn}\cdot - \mathbf{n}\cdot\mathbf{B})\,\mathbf{v}' &= \ldots, \end{aligned}$$

where $\ldots$ indicate the known derivatives w.r.t. η, ζ, and τ. Note, after minor algebra (dimension considerations) we arrive to the algebra of symmetric rep. of primitive variables, where $(\mathbf{n}, u) \rightleftharpoons (\mathbf{k}, \omega)$.

Characteristics in iMHD

- Choose **B** and **n** to be of the form

$$\mathbf{B} = (0, 0, B)\,, \qquad \mathbf{n} = (n_x, 0, n_z)$$

and introduce the *Alfvén speed* and *sound speed*,

$$b \equiv B/\sqrt{\rho}, \qquad c \equiv \sqrt{\gamma p/\rho}$$

Characteristics in iMHD

$$
\begin{pmatrix}
-\gamma u & n_x c & 0 & n_z c & 0 & 0 & 0 & 0 \\
n_x c & -u & 0 & 0 & n_x c & -n_z b & 0 & n_x b \\
0 & 0 & -u & 0 & 0 & 0 & -n_z b & 0 \\
n_z c & 0 & 0 & -u & n_z c & 0 & 0 & 0 \\
0 & n_x c & 0 & n_z c & -\frac{\gamma}{\gamma-1} u & 0 & 0 & 0 \\
0 & -n_z b & 0 & 0 & 0 & -u & 0 & 0 \\
0 & 0 & -n_z b & 0 & 0 & 0 & -u & 0 \\
0 & n_x b & 0 & 0 & 0 & 0 & 0 & -u
\end{pmatrix}
\begin{pmatrix}
\frac{c}{\gamma\rho}\rho' \\
v_x' \\
v_y' \\
v_z' \\
\frac{\gamma-1}{c} e' \\
\frac{1}{\sqrt{\rho}} B_x' \\
\frac{1}{\sqrt{\rho}} B_y' \\
\frac{1}{\sqrt{\rho}} B_z'
\end{pmatrix}
= \dots
$$

where $n_z = B_n/B$, and, $n_x = [1 - (B_n/B)^2]^{1/2}$. This is the EVP derived before for MHD waves. Further, if algebra is the same, physics must be probably he same too!

Characteristics in iMHD

• Characteristics are obtained when det[LHS] $= 0$, so that the full inhomogeneous problem cannot be solved. The solutions cannot be propagated away from the manifold $\xi = \xi_0$. This condition may be written as

$$\Delta = \frac{\gamma^2}{\gamma - 1} u^2 \left(u^2 - {b_n}^2\right) \left[u^4 - \left(b^2 + c^2\right) u^2 + {b_n}^2 c^2\right] = 0$$

where $b_n \equiv \mathbf{n} \cdot \mathbf{B}/\sqrt{\rho}$. We recovered the DE for linear MHD waves in the disguise of an equation for the characteristic speeds u!

Characteristics in iMHD

• We immediately conclude that a spurious root $u = 0$ was introduced by $u\mathbf{n} \cdot \mathbf{B}'$.
• Seven real characteristics are obtained.
• The matrix on LHS is real, symmetry and has real EVs.
• $\Rightarrow$ The eqs of iMHD are symmetric hyperbolic PDEs and the IVP is well-posed.
• Disregarding redundant root, the characteristic speeds are:

$$u = u_E \equiv 0,$$

$$u = u_A \equiv \pm b_n,$$

$$u = u_s \equiv \pm \left[\frac{1}{2} \left(b^2 + c^2 \right) - \frac{1}{2} \sqrt{ \left(b^2 + c^2 \right)^2 - 4 {b_n}^2 c^2 } \right]^{\frac{1}{2}},$$

$$u = u_A \equiv \pm \left[\frac{1}{2} \left(b^2 + c^2 \right) + \frac{1}{2} \sqrt{ \left(b^2 + c^2 \right)^2 - 4 {b_n}^2 c^2 } \right]^{\frac{1}{2}},$$

Characteristics in iMHD

- The characteristic speeds are ordered:

$$0 = |u_E| \leq |u_s| \leq |u_A| \leq |u_f| < \infty$$

- Degeneracies occur for
  $\mathbf{n} \parallel \mathbf{B}$:

$$|u_s| = \min(b, c), \qquad |u_A| = b, \qquad |u_f| \max(b, c)$$

and for $\mathbf{n} \perp \mathbf{B}$:

$$|u_s| = |u_A| = 0, \qquad |u_f| = \left(b^2 + c^2\right)^{\frac{1}{2}}.$$

Characteristics in iMHD

Entropy characteristics ($u = 0$)

$$S'/S = -\gamma\rho'/\rho = \gamma e'/e \neq 0, \qquad \mathbf{v}' = 0, \qquad \mathbf{B}' = 0$$

Alfvén characteristics ($u = u_A$)

$$B'_y = -\sqrt{\rho}v'_y \neq 0,\ \rho' = 0,\ e' = 0,\ v'_x = v'_z = 0,\ B'_x = B'_z = 0$$

Magneto-acoustic characteristics ($u = u_{s,f}$)

$$v'_x = \frac{n_x}{n_z}\frac{u^2}{u^2-b^2}v'_z \neq 0, \qquad B'_z = -\frac{n_x}{n_z}B'_x = \frac{n_x b}{u}\sqrt{\rho}v'_x \neq 0$$

$$\rho' = \frac{\gamma\rho}{c^2}e' = \frac{\rho u}{n_z c^2}v'_z \neq 0, \qquad v'_y = 0, \qquad B'_y = 0$$

Reference Literature

C.M. Bender & S.A. Orszag: Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory
Springer Verlag, 1999, **ISBN: 0-387989315**
Advanced intro on applied maths. Excellent authority.
J. Billingham & A.C. King: Wave Motion
Cambridge Texts in Applied Mathematics, CUP, 2001, **ISBN: 0-521-634504 pbk**
Good summary of wave theory. Parts of the book may be advanced for UK maths PGs.
C.A.J. Fletcher: Computational Techniques for Fluid Dynamics: Vol. 1-2
Springer, 2004 (2nd edition), **ISBN: 3-540-53058-4 pbk**
Advanced on CFD, covering all essentials. Excellent insight.
J.P. Goedbloed & S Poedts: Principles of Magnetohydrodynamics: Vol. 1-2
CUP, 2004, **ISBN: 0-521-62347-2 pbk**
Advanced on MHD waves, instabilities and spectral theory. Excellent insight.
R. Knobel: An Introduction to the Mathematical Theory of Waves
American Mathematical Society, 2000, **ISBN: 0-8218-2039-7 pbk**
Basic though excellent intro material on hyperbolic waves.
Sir J. Lighthill: Waves in Fluids
Cambridge Mathematical Library, CUP, 2003, **ISBN: 0-521-01045 pbk**
Covers physics part on fluid waves. Excellent insight.
K. Srinivas & C.A.J. Fletcher: Computational Techniques for Fluid Dynamics: A Solutions Manual
Scientific Computation, Springer-Verlag Berlin and Heidelberg GmbH & Co., **ISBN: 978-354054304-X pbk**
Advanced on MHD waves, instabilities and spectral theory. Excellent insight
G.B. Whitham: Linear and Nonlinear Waves
Pure and Applied Mathematics, Wiley, 1999, **ISBN: 0-471-35942 pbk**
Covers all the theory. Really a text for graduate and research students. Authoritative reference.

Conservation, Shocks and Characteristics in MHD

The End