(**1**) Show that

$$S_1 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \sum_{n=1}^{\infty} n^{-2}$$

satisfies

$$S_1 = (1 + \frac{1}{3^2} + \frac{1}{5^2} + ...) + \frac{1}{4} S_1.$$

Using Eq. (2.23) in the Notes, deduce that $S_1 = \pi^2/6$.
Using a similar method, show that

$$\sum_{n=1}^{\infty} = \frac{\sin^2 \frac{n\pi}{3}}{n^2} = \frac{\pi^2}{9}.$$

(**2**) A uniform string of length L, for which the speed of transverse waves is c and the mass per unit length is ρ, is fixed at its ends $x = 0$ and $x = L$. The point $x = L/3$ is drawn aside a distance h, and the string is then released from rest. Assuming that the displacement is given by

$$y(x,t) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right),$$

find the constant $\{\alpha_n\}$.

Find the energy in each of the normal modes. Use the last result in Q1 in this Example Sheet to verify that the total energy is equal to the work done in displacing the string originally.

(**3**) Generalise Q2 to the case when the point $x = b$ $(0 < b < L)$ is drawn aside a distance h, and the string is released from rest. From your results for $\{\alpha_n\}$, deduce (by considering the initial configuration) that

$$\sum_{n=1}^{\infty} \frac{\sin^2 \frac{n\pi b}{L}}{n^2} = \frac{b(L-b)\pi^2}{2L^2}.$$

(**4**) By writing $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$ etc., show that

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta. \qquad (1)$$

The transverse displacement of a uniform string of length L and wave speed c is $y(x,t)$. Given that $y(0,t) = y(L,t) = 0$ for $\forall\, t$, and that $y(x,0) = 0$, $\dot{y}(x,t) = V\sin^3(\pi x/L)$, find $y(x,t)$. [You may assume that

$$y(x,t) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right),$$

and it is suggested that you use Eq. (1) to find the constant β_n by *inspection*].

(**5**) A uniform string of line density ρ is under a tension ρc^2 and is fixed at the ends $x = 0$ and $x = L$. At time $t = 0$, the string is in its equilibrium position and the velocity of the point x is V for $d - a < x < d + a$, and 0 for all other x between $x = 0$ and $x = L$. Show that, at any subsequent time t, the transverse displacement $y(x,t)$ of the string is given by

$$y(x,t) = \frac{4VL}{c\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{\sin\left(\frac{n\pi d}{L}\right) \sin\left(\frac{n\pi a}{L}\right)}{n^2} \right\} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right).$$

(**6**) A string of length L and line density ρ is under a tension ρc^2. Its transverse displacement is $y(x,t)$. The string is fastened to rings at $x = 0$ and $x = L$. The masses of the rings can be neglected, and they slide on two smooth fixed wires at $x = 0$ and $x = L$, one ring being on each wire; the wires are parallel to $0y$. By considering the equations of motion of the rings, show that

$$\frac{\partial y}{\partial x} = 0$$

at $x = 0$ and $x = L$ for $\forall\ t$. By seeking separable solutions, find the normal modes of vibration.

(**7**) Consider the problem (*a forced oscillation*):

$$u_{tt} = c^2 u_{xx} \quad (0 \le x \le L, t \ge 0)$$

with $u(0,t) = 0$, $u(L,t) = a\sin\omega t$, $u(x,0) = \dot{u}(x,0) = 0$, where a and ω are positive constants with ω *not* being a multiple of $\pi c/L$. Write

$$u(x,t) = v(x,t) + \frac{a\sin(\omega x/c)\sin(\omega t)}{\sin(\omega L/c)}$$

and formulate the problem in terms of $v(x,t)$. Hence show that

$$u(x,t) = a\left\{\frac{\sin\left(\frac{\omega x}{c}\right)}{\sin\left(\frac{\omega L}{c}\right)}\right\}\sin\omega t + \frac{2a\omega c}{L}\sum_{p=1}^{\infty}\frac{(-1)^p \sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{p\pi ct}{L}\right)}{\left(\frac{p\pi c}{L}\right)^2 - \omega^2}.$$

Now suppose that ω approaches $(\pi c/L)$. By writing $\omega = (\pi c/L)(1+\epsilon)$ and considering the limit as $\epsilon \to 0$, show that when $\omega = \pi c/L$,

$$u(x,t) = u_\star(x,t) + \frac{2a}{\pi}\sum_{p=2}^{\infty}\frac{(-1)^p \sin\left(\frac{p\pi x}{L}\right)\sin\left(\frac{p\pi ct}{L}\right)}{p^2 - 1},$$

where $u_\star(x,t)$ is to be found.

(The last part of this question is an illustration of *resonance* where the forcing term is at a natural frequency. To avoid possible resonance, soldiers do *not* march in step across, for example, a bridge.)