Linear and nonlinear waves and oscillations **Example Sheet 3**

(**1**) Solve the ODE

$$\ddot{y} + n^2 y = f \cos \omega t,$$

with $y(0) = \dot{y}(0) = 0$, where f, n, and ω are positive constants, when

$$(i)\ \omega \neq n, \qquad (ii)\ \omega = n.$$

Verify that the limit of your answer to (i) as $\omega \to n$ for fixed t is the same as you answer to (ii).

(**2**) Solve the ODE

$$\ddot{y} + 2\lambda\dot{y} + n^2 y = f \cos \omega t,$$

with $y(0) = \dot{y}(0) = 0$, where f, n, ω and λ are positive constants with $\lambda \ll n$.

(i) Verify that your result is consistent with results in *Q. 1* when

$$(a)\ \lambda = 0 \text{ and } \omega \neq n; \qquad (b)\ \omega = n \text{ and } \lambda \to 0.$$

(ii) Show that for large λt, $y \approx \rho \cos(\omega t - \alpha)$, where ρ and α are constants. Sketch the graph of ρ against ω for fixed n.

(**3**) Write $\rho = \rho_0(1 + s)$ and $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, $w = \partial\phi/\partial z$, [see Eq. (3.13)]. Show that Eq. (3.9) becomes

$$\frac{\partial s}{\partial t} = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right).$$

Given that conditions are steady as $|\mathbf{x}| \to \infty$, deduce from Eq. (3.11) that

$$c^2 s = -\frac{\partial\phi}{\partial t},$$

and hence show that

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right). \qquad (A)$$

This is the 3D form of the wave equation.

(**4**) (*Spherically symmetric sound waves.*) Suppose $\phi = \phi(r, t)$ where (as usual) $r^2 = x^2 + y^2 + z^2$ (i.e. ϕ depends on x, y, z only via their involvement in r). Show successively that

$$\frac{\partial r}{\partial x} = r^{-1}x, \quad \frac{\partial\phi}{\partial x} = r^{-1}\frac{\partial\phi}{\partial r}x, \quad \frac{\partial^2\phi}{\partial x^2} = r^{-1}\frac{\partial}{\partial r}\left(r^{-1}\frac{\partial\phi}{\partial r}\right)x^2 + r^{-1}\frac{\partial\phi}{\partial r}$$

Given that ϕ satisfies the wave equation Eq. (3.14), i.e. (A) in *Q. 3* above, deduce that

$$c^2\left(\phi_{rr} + 2r^{-1}\phi_r\right) = \phi_{tt}. \qquad (B)$$

Show that (B) has solutions of the form

$$\phi = Ar^{-1}\exp\left\{\frac{i\omega}{c}(r \pm ct)\right\},$$

where ω is a real constant and A is a complex constant.

(**5**) (*A planar waveguide.*) Sound waves propagate in the positive Oz direction between the walls $x = 0$, $x = d$. The velocity potential ϕ satisfies $\phi_{xx} + \phi_{zz} = \phi_{tt}/c^2$. Seek solutions of the form $\phi = f(x)\exp[i(\omega t - kz)]$, where ω and k are real positive constants. Show that the solution satisfying $\phi_x = 0$ at $x = 0, d$ (no velocity perpendicular to the walls) can be written $\phi = 2A\cos(n\pi x/d)\exp[i(\omega t - kz)]$, where A is an arbitrary (complex) constant, and

$$k = k_n = \left(\frac{\omega^2}{c^2} - \frac{n^2\pi^2}{d^2}\right)^{\frac{1}{2}}, \qquad (n = 0, 1, 2...).$$

(**6**) (*A circular waveguide. Model of a stethoscope.*) Sound waves propagate in the positive Oz direction inside the circular cylinder $r = a$, where $r^2 = x^2 + y^2$. The velocity potential ϕ satisfies $c^2(\phi_{xx} + \phi_{yy} + \phi_{zz}) = \phi_{tt}$, and $\phi = g(r)\exp[i(\omega t - kx)]$, where ω and k are real constants. Show that

$$g''(r) + \frac{1}{r}g'(r) + m^2 g(r) = 0, \qquad m^2 = \frac{\omega^2}{c^2} - k^2.$$

[You may assume that $m^2 \geq 0$ for reasons analogous to those in the answer to *Q. 5* above.] Given that ϕ is bounded at $r = 0$ and that $\partial\phi/\partial r = 0$ at $r = a$, show that $g(r) \propto J_0(\beta_n r/a)$, where β_n is the n^{th} non-zero root of $J_0'(x) = 0$ with $\beta_0 = 0$ and

$$k = k_n = \left(\frac{\omega^2}{c^2} - \frac{n^2\pi^2}{a^2}\right)^{\frac{1}{2}}.$$

[Here $J_0(x)$ is the Bessel function of order zero - see Eq. (2.30) and handout].

(**7**) (*An organ pipe; a flute*) Consider a sound wave in a straight tube $0 \leq x \leq L$. The velocity potential $\phi = \phi(x, t)$, where

$$\phi_{tt} = c^2\phi_{xx}.$$

At $x = 0$, the tube is closed and $\phi_x = 0$ there. At $x = L$, the tube is open to the atmosphere and $\phi_t = 0$ there.[1] Show that the normal modes are given by

$$\phi = \phi_n \propto \cos\left(\frac{\omega_n x}{c}\right)\cos(\omega_n t + \epsilon_n), \quad \omega_n = \frac{\pi c}{L}\left(n + \frac{1}{2}\right).$$

[HINT: Find the normal modes by looking for separable solutions.]

[1] Because $\rho = \rho_0$ = atmospheric density there and, thus, $s = 0$ where $\rho = \rho(1 + s)$. Hence - see *Q. 3* on this sheet - $\phi_t = 0$.