Linear and nonlinear waves and oscillations **Example Sheet 4**

(**1**) A boat which is anchored in deep water is observed to rise and fall 15 times in a minute due to the passage of a progressive surface wave. Show that the phase speed of this wave is about 14 mph.

(**2**) Consider a progressive surface wave in water of depth 4 m with $\eta_0 = 1.25 \times 10^{-1}$ m (using the notation in the Notes) and wavelength $\lambda = 10$ m. The lateral extent L of the wave is 50 m, the density ρ of the water is 10^3 kg m^{-3} and $g \approx 9.8$ m s^{-2}.

Show that the mean energy per unit length in the direction of propagation is $\rho g \eta_0^2 L/2$, and evaluate this.

Find the phase speed c of this wave.

A Ford Fiesta has mass 1.2×10^3 kg and length 3 m. Find it's kinetic energy per unit length when travelling at speed c.

(**3**) In this question we look for *standing waves* with $\eta = \eta_0 \sin kx \cos \omega t$. The velocity potential ϕ must again satisfy Eqs. (4.8), (4.9), (4.10) and (4.11). Show that an appropriate form for ϕ is

$$\phi = f(z) \sin kx \sin \omega t,$$

and find $f(z)$. Show also that the dispersion relations Eq. (4.16) still hold.

Find the particle paths, and verify that each particle moves in a straight line.

(**4**) Short surface waves are affected by a phenomenon known as *surface tension.* This results in the pressure at the free surface not being continuous; Eq. (4.11) is consequently replaced by

$$\frac{\partial \phi}{\partial t} + g\eta = \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} \quad \text{at} \quad z = 0, \qquad \text{(A)}$$

where T is the magnitude of the surface tension. Show that for progressive waves on water of depth h with $\eta = \eta_0 \sin(kx - \omega t)$, $\phi = f(z) \cos(kx - \omega t)$, Eqs. (4.8), (4.9), (4.10) and Eq. (A) give the dispersion relation

$$\omega^2 = gk \tanh kh \left(1 + \frac{Tk^2}{\rho g}\right).$$

In the remainder of this question, suppose kh is large enough so that $\tanh kh \approx 1$, and

$$\omega^2 = gk \left(1 + \frac{Tk^2}{\rho g}\right))$$

Show that the phase velocity c and the group velocity c_g satisfy

$$c = \left(\frac{gT}{\rho}\right)^{\frac{1}{4}} \left(\frac{1 + x^2}{x}\right)^{\frac{1}{2}}, \qquad c_g = \frac{1}{2}c\left(\frac{1 + 3x^2}{1 + x^2}\right) \quad \text{with} \quad x^2 = \frac{Tk^2}{\rho g}.$$

Deduce that c has a minimum c_m when $x = 1$. Show that $c_g > c$ for $x > 1$ and that $c_g < c$ for $x < 1$. Sketch the graph of c against x.

Evaluate c_m for water with $T \approx 7.4 \times 10^{-2}$ kg s^{-2}, $\rho \approx 10^3$ kg m^{-3}, $g \approx 9.8$ m s^{-2}.

(**5**) Consider a fixed obstacle in a stream of constant speed U. The obstacle penetrates the surface and generates waves of the type investigated in the second part of *Q. 4* to which reference shall be made.

(*i*) Explain why the phase speed $c = c(k)$ of waves that appear steady when viewed from the obstacle must satisfy $c(k) = U$.

(*ii*) Provided $U > c_{min}$, there are two possible values of k. By considering the group velocity c_g show that the *shorter* waves (or *ripples* or *capillary* waves) are observed *upstream* of the obstacle.

(*iii*) What happens if $U < c_{min}$?

(**6**) An observer is moving with velocity v away from a stationary source of sound waves.

By imposing a velocity $-v$ on the whole motion (observer, source, waves), show that when the source generates waves of frequency ν, they are registered by the observer as having frequency $\nu(c - v)/c$.

Obtain the corresponding result when both source and observer are moving in the same direction with velocities u and v respectively.

An observer at rest notices that the frequency of the sound waves from a car appears to drop from 272 to 256 per second as the car passes her.

Show that the speed of the car is $c/33$.

How fast must she travel towards the car for the apparent frequency to rise to 280 per second?