(**1**) Show that the solution of

$$yz_x - 2xyz_y = 2xz$$

with $z = y^3$ on $x = 0$, $1 \le y \le 2$ is

$$z = \frac{(y + x^2)^4}{y}.$$

On what domain in the x-y plane does this solution apply?

(**2**) Integrate the associated equations to show that the solution of

$$x^3 \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$$

with $u = 1/(1 + x^2)$ on $y = 0$, $-\infty < x < \infty$, is

$$u = \frac{1 - 2x^2 y}{1 + x^2 - 2x^2 y}.$$

Explain why the solution is not defined when $y \ge 1/(2x^2)$.

(**3**) The density $\rho(x, y)$ of cars satisfies

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad \text{and} \quad \rho(x, 0) = f(x).$$

Show that on the straight line characteristics $x = \xi + F(\xi)t$, where $F(\xi) = c\{f(\xi)\}$,

$$\frac{\partial \xi}{\partial t} = -\frac{F(\xi)}{1 + F'(\xi)t}, \qquad \frac{\partial \xi}{\partial x} = \frac{1}{1 + F'(\xi)t}.$$

(*a*) Hence verify that $\rho = f(\xi)$ on the characteristics.

(*b*) Show that the solution breaks down for $t > 0$ if there is a ξ for which $F'(\xi) < 0$.

(**4**) Data taken on the M25 in 1999 suggests that the formula $v(\rho) = (V/P)(P - \rho)$, for $0 \le \rho \le P$, is a reasonable fit to the data with $V \approx 82$ mph. Estimate P in cars per mile, given that the observed maximum flowrate is $q_m \approx 7400$ cars per hour. Also, estimate the value of $c(\rho) = d(\rho v)/d\rho$ when $\rho \approx 200$ cars per mile.

(**5**) Given that $c = c(\rho)$ and that $\rho_t + c(\rho)\rho_x = 0$, show that

$$c_t + cc_x = 0.$$

If $\rho = \rho(x, 0) = f(x)$ at $t = 0$, deduce that in regions where $c(x, t)$ is continuously differentiable

$$c = c\{f(\xi)\} = F(\xi) \quad \text{on straight lines} \quad x = \xi + F(\xi)t.$$

A shock occurs with values of ρ on either side of the shock being equal to ρ_1 and ρ_2. Show that the velocity U of the shock is equal to

$$U = \frac{1}{2}\{c_1 + c_2\},$$

where $c_1 = c(\rho_1)$, $c_2 = c(\rho_2)$, in the following two cases:

(a) exactly when $q(\rho)$ is a quadratic function of ρ (b) approximately when the shock is weak, i.e., $|\rho_2 - \rho_1| \ll (\rho_1 + \rho_2)$.

[N.B. Recall that $c(\rho) = q'(\rho)$.]

(**6**) The flow of traffic is modelled with

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\{\rho v(\rho)\} = 0,$$

with $v(\rho) = (V/P)(P - \rho)$, where V and P are constants. Given that

$$\rho(x, 0) = \begin{cases} 0, & x \leq 0 \\ \rho_R(x^2/L^2), & 0 \leq x \leq L \\ \rho_R, & x \geq L \end{cases}$$

where L and ρ_R are constants with $\rho_R < P$, obtain the solution. Determine when and where the solution first breaks down.