KdV equation/waves of permanent form

1. (a) Show that the *KdV equation*,

$$u_t - 6uu_x + u_{xxx} = 0,$$

is invariant under the one-parameter continuous group of transformations

$$x' = kx, \quad t' = k^3 t \quad \text{and} \quad u' = k^{-2}u.$$

(b) Assuming that there exists a solution of the form

$$u = -(3t)^{-2/3}U(\xi), \quad \xi = x/(3t)^{1/3},$$

show that

$$U''' + (6U - \xi)U' - 2U = 0,$$

where $U' = dU/d\xi$.

2. (a) Look for a solution of the *sine-Gordon equation*

$$u_{tt} - u_{xx} + \sin u = 0,$$

in the form $u = f(\xi)$, where $\xi = x - ct$ for some given constant c and deduce that

$$\frac{1}{2}\left(c^2 - 1\right)(f')^2 + 2\sin^2\left(\frac{1}{2}f\right) = A$$

for some constant A of integration.

(b) Hence show that if $A = 0$ and $0 < c^2 < 1$ the solution can only be a kink (or antikink) and that

$$\tan\left(\frac{1}{4}f\right) = \pm\exp\left[\pm\frac{(\xi - \xi_0)}{\sqrt{1 - c^2}}\right]$$

or

$$\tan\left(\frac{1}{4}f\right) = \pm\exp\left[\mp\frac{(\xi - \xi_0)}{\sqrt{1 - c^2}}\right]$$

for some real constant ξ_0. [Hint: use $\int df/\sin(f/2) = 2\log[\tan(f/4)] + \text{const.}$]

(c) It is given that a solution u of the sine-Gordon equation satisfies

$$\tan\left(\frac{1}{4}u\right) = c\frac{\sinh\left(\frac{x}{\sqrt{1-c^2}}\right)}{\cosh\left(\frac{ct}{\sqrt{1-c^2}}\right)}.$$

Evaluate this solution for $0 < c^2 < 1$ asymptotically as $t \to \pm\infty$ and interpret the solution as an interaction of kinks. Your interpretation should include a qualitative description of the evolution of the solution.

3. (a) Look for a solution of the *nonlinear Schrödinger equation*

$$iu_t + u_{xx} + u\,|u|^2 = 0,$$

in the form

$$u = re^{i(\theta+\nu t)},$$

where $r = r(\xi)$, $\theta = \theta(\xi)$, $\xi = x - ct$ and ν is a real constant, and show that

$$\theta' = \frac{1}{2}\,(c + A/S) \quad \text{and} \quad (S')^2 = -2F(S),$$

with $S = r^2$ and $F(S) = S^3 - 2(\nu - \frac{1}{4}c^2)S^2 + BS + \frac{1}{2}A^2$, where A and B arbitrary (real) constants of integration;

(b) examine the nature of the zeros of the cubic F, and hence (briefly) discuss the occurrence and properties of periodic and solitons solutions for u;

(c) show that there exists solitary-wave solutions of the form

$$u(x,t) = ae^{i\left[\frac{1}{2}c(x-ct)+\nu t\right]}\,\mathrm{sech}\left[\frac{a\,(x-ct-x_0)}{\sqrt{2}}\right]$$

for all $a^2 = 2\left(n - \frac{1}{4}c^2\right) > 0$.

4. Consider the *RWL equation*, namely

$$u_t - uu_x - u_{xxt} = 0.$$

(a) Seek wave solutions of permanent form, $u(x,t) = f(\xi)$, where $\xi = x - ct$, showing that

$$c(f')^2 = \frac{1}{3}f^3 + cf^2 + Af + B,$$

for some constants A and B;

(b) show how the properties of the zeros of the cubic determine the existence and range of periodic solutions f;

(c) find all solutions which describe solitons.