The scattering problem

1. (a) Find the eigenvalues and eigenfunctions of

$$\psi'' + (\lambda - u(x))\psi, \tag{1}$$

when $u(x)$ is the step potential

$$u(x) = \begin{cases} 0 & \text{if } x < 0 \\ V & \text{if } x > 0, \end{cases} \tag{2}$$

with $V > 0$.

(b) Show, in particular, that there is a continuum eigenfunction, no discrete eigenfunction, and an eigenfunction which decays as $x \to \infty$ but is oscillatory in $x < 0$.

2. (Difficult) Consider the Schrödinger equation

$$\psi'' + (\lambda - u(x))\psi, \tag{3}$$

with $u(x)$ given by

$$u(x) = \begin{cases} -V & \text{if } -b \leq x \leq b \\ 0 & \text{if } x < -b \text{ or } x > b, \end{cases} \tag{4}$$

with $V > 0$.

(a) Evaluate the eigenvalues and eigenfunctions of the discrete spectrum;

(b) find the transmission and reflection coefficients.

[Hint: Because the potential is piecewise constant, the solutions are linear combinations of exponentials (real or complex). Use the requirements that ψ and ψ' must be continuous at the discontinuities of $u(x)$ to find a system of linear equations which expresses conditions on the coefficients of such exponentials and on the eigenvalues.]

3. (a) Evaluate

$$w = |a(k)|^2 + |b(k)|^2\,, \tag{5}$$

where $a(k)$ and $b(k)$ are defined by

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \to \infty \\ a(k)e^{-ikx} & \text{as } x \to -\infty \end{cases} \tag{6}$$

in the two previous exercises. Are they both equal to one?

(b) In quantum mechanics the probability flux is defined by $j = \mathrm{Im}\,(\psi^*\psi')$. Show that if we define the transmission coefficient τ as the ratio of the transmitted flux with respect to the incident flux, and the reflection coefficient as the ratio of the reflected flux with respect to the incident flux, then

$$\tau + \rho = 1 \tag{7}$$

in both cases.

(c) Why does this distinction not count in the second problem?