Solitons **Problems sheet 3**

The inverse scattering problem

1. Consider the Scrödinger equation

$$\psi'' + (\lambda - u(x))\psi = 0. \tag{1}$$

Suppose that $u(x)$ is reflectionless and that we are given the discrete spectra $\lambda_n = -\kappa_n^2$ and eigenfunctions $\psi_n(x)$, $n = 1, \ldots, N$ where N is finite. Moreover, we have

$$\psi_1(x) \sim c_1 \exp(-\kappa_1 x), \ldots, \psi_N(x) \sim c_N \exp(-\kappa_N x) \quad \text{as } x \to \infty. \tag{2}$$

(a) Determine the two N-vectors $E(x)$ and $B(x)$ and the $N \times N$-matrix $A(x)$ such that

$$K(x,x) = -E^T(x)A^{-1}(x)B(x), \tag{3}$$

where $K(x,x)$ is the solution of the Marchenko equation

$$K(x,z) + F(x+z) + \int_x^\infty K(x,y)F(y+z)dy = 0, \quad z > x > -\infty \tag{4}$$

and

$$F(X) = \sum_{n=1}^{N} c_n^2 \exp\left(-\kappa_n X\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikX}dk; \tag{5}$$

(b) prove that

$$u(x) = -2\frac{d^2}{dx^2}\log\det A. \tag{6}$$

[Hint: Use the formula

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A}\frac{\partial \det A}{\partial A_{ji}}. \tag{7}$$

(Note the order of the indices!)]

2. Consider the solution of the Schrödinger equation

$$\psi'' + (\lambda - u(x))\,\psi = 0 \tag{8}$$

given by

$$\psi_{\text{-}}(x) = e^{-ikx} + \int_{-\infty}^{x} L(x,z)e^{-ikz}dz, \tag{9}$$

with $L(x,z) = 0$ for $z > x$ and $L(x,z), L_z(x,z) \to 0$ for $z \to -\infty$. Prove that

$$L_{xx}(x,z) - L_{zz}(x,z) - u(x)L(x,z) = 0 \quad \text{for } z < x \tag{10}$$

and

$$u(x) = 2\frac{dL(x,x)}{dx}. \tag{11}$$

[Hint: Use integration by parts to show that

$$\psi_{\text{-}}(x) = e^{-ikx}\left(1 + \frac{iL(x,x)}{k} + \frac{L_z(x,x)}{k^2}\right) - \frac{1}{k^2}\int_{-\infty}^{x} L_{zz}(x,z)e^{-ikz}dz.] \tag{12}$$