Solitons **Problem sheet 4**

The initial-value problem for the KdV equation

1. Consider the initial-value problem for the KdV equation with $u(x,0) = \alpha f(x)$, with $\alpha > 0^+$ and $0 \leq \int_{-\infty}^{\infty} f(x)dx < \infty$. It can be proved that under these hypothesis no discrete spectrum exists.

   (a) Show that in the limit $\alpha \to 0$ the reflection and transmission coefficients of the direct scattering problem are given by

   $$a(k;\alpha) \sim 1 - \frac{i\alpha}{2k}\int_{-\infty}^{\infty} f(x)dx, \quad \alpha \to 0^+ \tag{1}$$

   and

   $$b(k;\alpha) \sim -\frac{i\alpha}{2k}\int_{-\infty}^{\infty} e^{-2ik\xi} f(\xi)d\xi, \quad \alpha \to 0^+, \tag{2}$$

   fore each fixed k. [Hint: use direct asymptotic expansion for the solution ψ of the Schrödinger equation. Use also the formula for a particular integral $g(x)$ of an inhomogeneous linear differential equation, $\psi'' + a_1(x)\psi' + a_0(x)\psi = h(x)$: $g(x) = \int_{x_0}^{x} r(x,\xi)h(\xi)d\xi$, where

   $$r(x,\xi) = 1/w(\xi)\begin{vmatrix} u_1(\xi) & u_2(\xi) \\ u_1(x) & u_2(x) \end{vmatrix}. \tag{3}$$

   Here $w(x)$ is the Wronskian and u_1 and u_2 are two linear independent solutions of the homogeneous equation.]

   (b) Using the inverse scattering transform, show that, as $\alpha \to 0^+$, the first approximation to the solution for u corresponds to the solution obtained by using the Fourier transform to solve the linear problem

   $$v_t + v_{xxx} = 0, \quad v(x,0) = \alpha f(x), \quad -\infty < x < \infty. \tag{4}$$

2. Consider the initial-value problem of the KdV equation given by

   $$u(x) = \begin{cases} -V & \text{if } -b \leq x \leq b \\ 0 & \text{if } x < -b \text{ or } x > b, \end{cases} \tag{5}$$

   (see problem sheet 2, exercise 2).

   (a) Give a brief qualitative description of the time-evolution of the solution for $t > 0$ and various V. If it helps, you may use diagrams.

   (b) Using the inverse scattering theory, find the solution in the limit $t \to \infty$, with the condition $Vb^2 < \pi^2/4$