Solitons — Problem sheet 5

The Lax formalism

1. The self-adjoint of an operator A, acting in some Hilbert space $\mathcal{H}$, is defined as the operator $A^\dagger$ such that

$$(A^\dagger u, v) = (u, Av), \quad \forall u, v \in \mathcal{H}. \tag{1}$$

Let H be and U be a Hermitian and unitary operators respectively; this means that

$$(u, Hv) = (Hu, v) \quad \text{and} \quad (Uu, Uv) = (u, v), \quad \forall u, v \in \mathcal{H}, \tag{2}$$

or equivalently

$$H^\dagger = H \quad \text{and} \quad U^\dagger U = I, \tag{3}$$

where I is the identity operator. It can be proved that there exist always unitary transformations W and Y such that $\bar{H} = W^\dagger H W$ and $\bar{U} = Y^\dagger U Y$ are diagonal operators.

(a) Prove that $\bar{H}$ is Hermitian and $\bar{U}$ unitary;

(b) prove that the eigenvalues of an Hermitian operator are always real;

(c) prove that if λ_1 and λ_2 are two eigenvalues of H such that $\lambda_1 \neq \lambda_2$ then the corresponding eigenvectors u_1 and u_2 are orthogonal, *i.e* $(u_1, u_2) = 0$.

(d) Suppose that $U = U(t)$ is a function of time and that H is constant. Moreover, suppose that $[U, H] = 0$. If A_0 does not depend on time, let us define

$$A(t) = U^\dagger(t) A_0 U(t). \tag{4}$$

In addition, the following equation holds:

$$iU_t = HU. \tag{5}$$

Prove that

$$iA_t = [A(t), H]. \tag{6}$$

2. (a) Cast the system

$$\dot{x} = gy, \quad \dot{y} = -gx, \tag{7}$$

where $g(x, y, t)$ is a given continuous function, into the equivalent form

$$L_t + [L, M] = 0, \tag{8}$$

finding L as some symmetric, and M as some antisymmetric, real 2×2 matrix whose elements depend upon x, y and g;

(b) using the Lax theory, deduce that $x^2 + y^2$ is a constant of motion.