1 Waves on Strings

1.1 What is a "wave"?

Difficult to define precisely: here are two "definitions".

• **COULSON (1941)**: "We are all familiar with the idea of a wave; thus, when a pebble is dropped into a pond, water waves travel radially outwards; when a piano is played, the wires vibrate and sound waves spread throughout the room; when a radio station is transmitting, electric waves move through the ether. These are all examples of wave motion, and they have two important properties in common: firstly, **energy is propagated** to distant points; and secondly, the **disturbance travels** through the medium **without** giving the medium as a whole **any permanent displacement**."

• **WHITHAM (1974)**: "...but to cover the whole range of wave phenomena it seems preferable to be guided by the intuitive view that **a wave is any recognizable signal** that is **transferred** from one part of the medium to another **with a recognizable velocity of propagation**."

We begin with, perhaps, the simplest possible example.

1.2 Derivation of Governing PDE

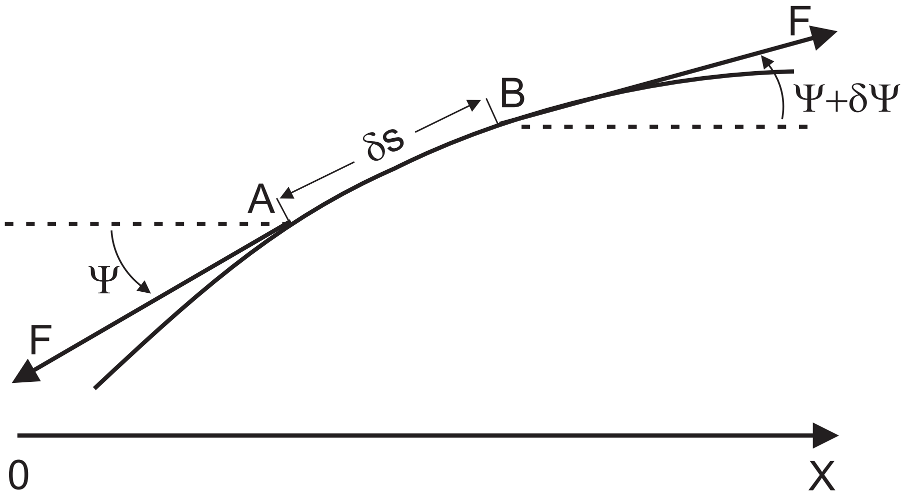

Figure 1: A piece S of a string

• We suppose the string is under tension F, and that its mass per unit length is ρ. We consider transverse motion only ($\perp$ Ox), and let the displacement be $y(x,t)$; we shall suppose y is small or -more precisely- we suppose $|\partial y/\partial x| \ll 1$ everywhere.

• Longitudinal motion negligible $\Rightarrow$ F is independent of x (see part ii below). We also take ρ independent of x.

• Apply N2 to a small element of the string AB of length δs.

$$\rho \delta s \frac{\partial^2 y}{\partial t^2} = F\{sin(\psi + \delta\psi) - sin\psi\}. \qquad (1)$$

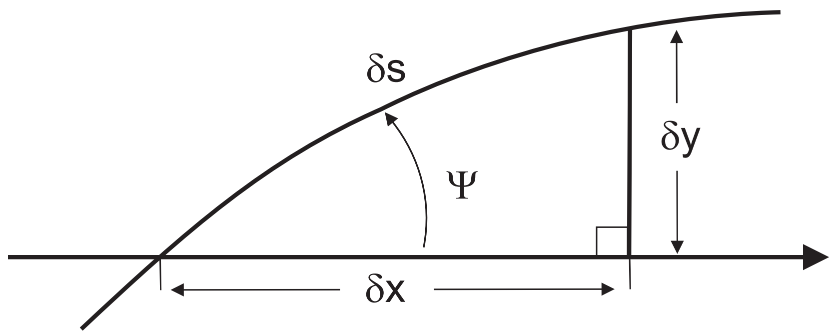

Figure 2: Local geometry of string S

Now, from sketch Fig. 2

$$\delta s^2 \approx \delta x^2 + \delta y^2 \Rightarrow \delta s \approx \left\{ 1 + \left(\frac{\partial y}{\partial x} \right)^2 \right\}^{1/2} \delta x \qquad (2)$$

Therefore, because $| \partial y/\partial x | \ll 1 \; \forall \; x$ (by assumption),

$$\delta s \approx \delta x \tag{3}$$

to highest order. Likewise

$$\tan\psi = \partial y/\partial x \ll 1 \Rightarrow \psi \approx \partial y/\partial x,$$

and, in Eq. (1),

$$\begin{aligned} \sin(\psi + \delta\psi) - \sin\psi &\approx \cos\psi \cdot \delta\psi \\ &\approx \{1 + \tan^2\psi\}^{-1/2}\delta\psi \\ &\approx \delta\psi \\ &\approx \delta(\partial y/\partial x) \\ &\approx (\partial^2 y/\partial x^2)\delta x. \end{aligned}$$

Thus Eq.(1) becomes

$$\frac{\partial^2 y}{\partial t^2} = \frac{F}{\rho}\frac{1}{\delta x}\frac{\partial^2 y}{\partial x^2}\delta x = \frac{F}{\rho}\frac{\partial^2 y}{\partial x^2}. \tag{4}$$

Finally we have

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{5}$$

where the constant c satisfies

$$c^2 = \frac{F}{\rho}. \tag{6}$$

• Eq. (5) is the 1D wave equation and c is the wave speed.

• (i) For the D string of a violin, $F \approx 55$ N, $\rho \approx 1.4\times 10^{-3}$ kgm^{-1} $\Rightarrow c \approx 200$ ms^{-1}

• (ii) We have assumed F is uniform. Hooke's Law $\Rightarrow$ change in $F \propto$ change in length. But

$$\begin{aligned}
\text{change in length} &= \delta s - \delta x \\
&\approx \left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}^{1/2} \delta x - \delta x \\
&\approx \left\{1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right\} \delta x \\
&= \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \delta x
\end{aligned}$$

which is second-order in small quantities $\Rightarrow$ the assumption of uniform F is OK.

• (*iii*) The kinetic energy (KE) of an element of length δs is

$$\frac{1}{2}\rho\delta s\left(\frac{\partial y}{\partial t}\right)^2 \approx \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \delta x,$$

which implies that the KE between $x = a$ and $x = b$ $(> a)$ is

$$\text{KE} = T = \frac{1}{2}\rho\int_a^b \left(\frac{\partial y}{\partial t}\right)^2 dx. \tag{7}$$

The potential energy (PE) of an element of length δs is

$$\begin{aligned} F \times \text{increase in length} &= F(\delta s - \delta x) \\ &\approx \frac{1}{2}F\left(\frac{\partial y}{\partial x}\right)^2 \delta x \quad (\text{from}(ii)). \end{aligned}$$

Thus the PE between $x = a$ and $x = b$ $(> a)$ is

$$\text{PE} = V = \frac{1}{2}F\int_a^b \left(\frac{\partial y}{\partial x}\right)^2 dx. \tag{8}$$

NB T, V are second-order in small quantities, i.e. $(\partial y/\partial x)^2$, $(\partial y/\partial t)^2$, whereas the wave equation Eq. (5) itself is first-order.

1.3 D'Alembert's solution and simple applications

• Unusually we can find the general solution of the wave equation Eq. (5). Change variables from (x, t) to (u, v), where

$$u = x - ct, \qquad v = x + ct. \tag{9}$$

Chain rule ⇒

$$\begin{aligned}\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} = y_u + y_v \Rightarrow \\ \frac{\partial^2 y}{\partial x^2} &= \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)(y_u + y_v) = y_{uu} + 2y_{uv} + y_{vv},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = -cy_u + cy_v \Rightarrow \\ \frac{\partial^2 y}{\partial t^2} &= c^2\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)(y_u - y_v) \\ &= c^2(y_{uu} - 2y_{uv} + y_{vv}).\end{aligned}$$

- Substitute in the wave equation Eq. (5)

$$c^2(y_{uu} + 2y_{uv} + y_{vv}) = c^2(y_{uu} - 2y_{uv} + y_{vv})$$

$\Rightarrow$

$$y_{uv} = \frac{\partial^2 y}{\partial u \partial v} = 0. \tag{10}$$

Therefore,

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right) = 0 \Rightarrow \frac{\partial y}{\partial v} = g_\star(v),$$

where $g_\star$ is any function[1] $\Rightarrow$

$$y = \underbrace{\int^v g_\star(s)ds}_{g(v)} + f(u),$$

where f is any function[1]. Thus

$$y = f(u) + g(v),$$

i.e.

$$y = f(x - ct) + g(x + ct). \tag{11}$$

Eq. (11) is d'Alembert's solution (the general solution) of the wave equation (5), first published in 1747 [J. le Rond d'Alembert (1717-83)].

[1] Of course f, g must be differentiable (except, perhaps, at isolated points)

• The functions f and g in Eq. (11) are determined by the boundary and initial conditions. For the moment we suppose the string is unbounded in both directions, i.e. $-\infty < x < \infty$.

To begin with, suppose that, at $t = 0$,

$$y(x,0) = \phi(x), \quad \dot{y}(x,0) = 0. \tag{12}$$

Thus the string is initially at rest $\forall\, x$, but has a displacement given by $y = \Phi(x)$.

From (11) and (12) we must have

$$f(x) + g(x) = \Phi(x), \quad -cf'(x) + cg'(x) = 0.$$

where $'$ denotes "derived function". The second gives $f'(x) = g'(x) \Rightarrow f(x) = g(x) + \alpha$, where α is a constant. The first then gives:

$$f(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\alpha, \quad g(x) = \frac{1}{2}\Phi(x) - \frac{1}{2}\alpha.$$

Thus, from Eq. (11):

$$y(x,t) = \frac{1}{2}\Phi(x - ct) + \frac{1}{2}\Phi(x + ct). \tag{13}$$

1.3.1 Examples

The Heaviside function

The Heaviside [O. Heaviside (1850-1925)] function $H(x)$ is defined by

$$H(x) = \begin{cases} 1 & (x \geq 0) \\ 0 & (x < 0) \end{cases} \tag{14}$$

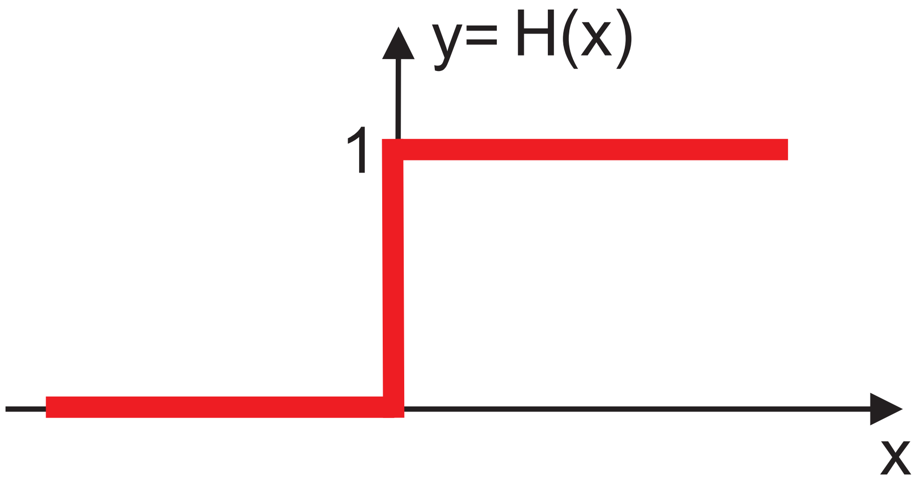

Figure 3: Heaviside function

Example 1

At $t = 0$, an infinite string is at rest and

$$y(x, 0) = b\{H(x + a) - H(x - a)\}, \qquad (15)$$

where $a, b > 0$ constants. Find $y(x, t)$ for $\forall\ x, t$ and sketch your solution.

Solution

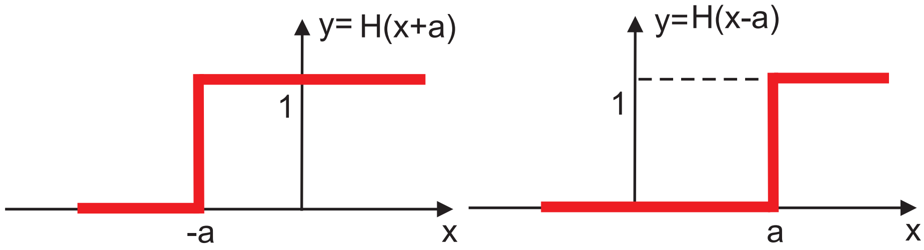

Figure 4: Shifted Heaviside functions

Thus Eq. (15) has the sketch $y(x, 0)$

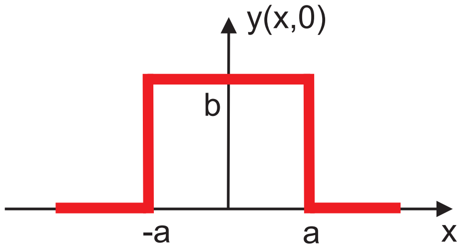

Figure 5: The initial profile $y(x, 0)$

Eq. (13) gives

$$\begin{aligned} y(x,t) &= \frac{b}{2}\{H(x-ct+a) - H(x-ct-a)\} \\ &+ \frac{b}{2}\{H(x+ct+a) - H(x+ct-a)\} \end{aligned} \tag{16}$$

The first term is like $y(x,0)$ except that

- (i) its height is $(1/2)b$, not b, and
- (ii) its end points are $(ct-a, ct+a)$, not $(-a, a)$.

This is a signal with graph like Fig. 5 except for (i) and (ii). Thus the first term in Eq. (16) has graph:

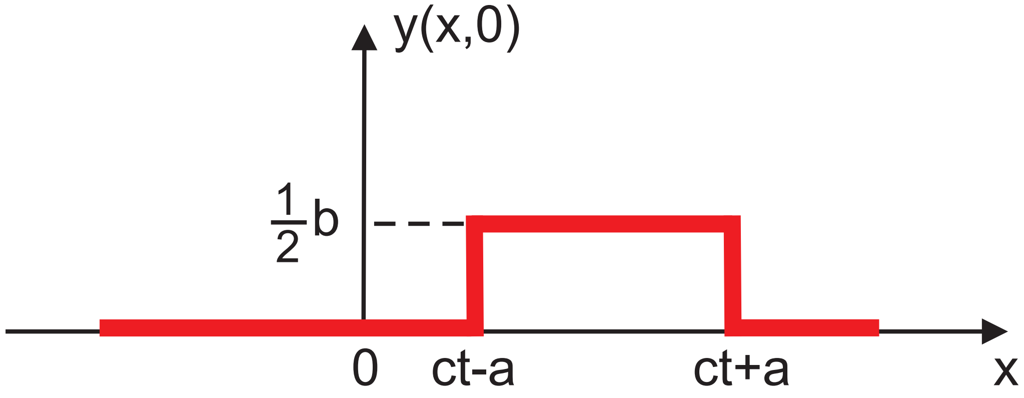

Figure 6: Travelling to right with speed c

Likewise the second term has graph:

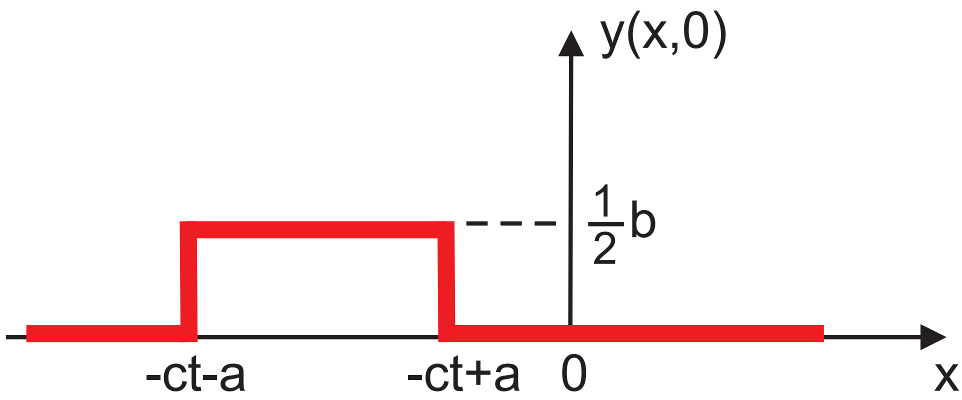

Figure 7: Travelling to left with speed c

The sum of the two pulses has a graph which depends on whether they overlap; this happens for t such that

$$-ct + a > ct - a$$

$\Rightarrow$

$$t < a/c.$$

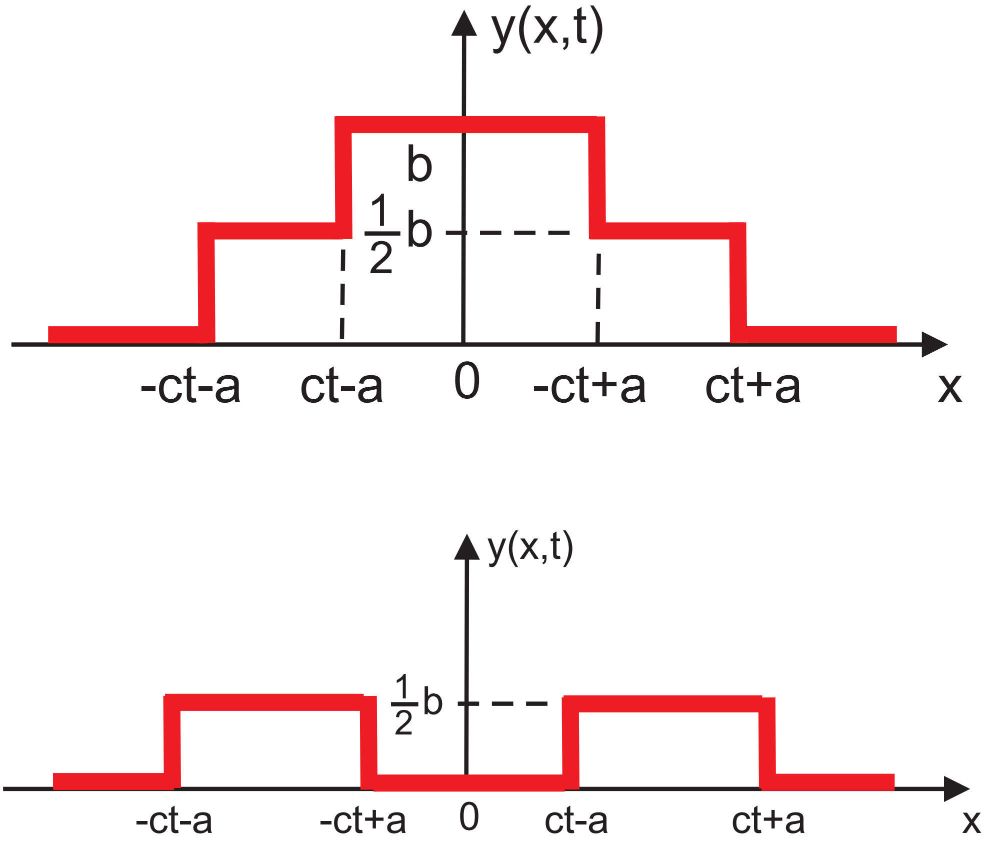

Figure 8: (a) $t < a/c$; (b) $t > a/c$

• This example illustrates well what Eq. (11) represents. The term $f(x-ct)$ has the same shape and size $\forall\, t$ (wave of permanent form); as t increases the profile moves to the right with speed c. Likewise $g(x+ct)$ is a profile of constant shape and size that moves to the left with speed c. Each is a travelling wave (or progressive wave). In the above example, the initial profile splits into two; one half travels to the right, one half to the left.

Example 2

Consider Eq. (12) with $\Phi(x) = a\sin(kx)$, where a and k are constants.

From Eq. (13) $\Rightarrow$

$$y(x,t) = \frac{1}{2}a\left\{\sin[k(x-ct)] + \sin[k(x+ct)]\right\}. \qquad (17)$$

We shall revisit Eq. (17) soon.

• More general than Eq. (12) is the case when the string is also moving at $t = 0$.

$$y(x,0) = \Phi(x), \qquad y_t(x,0) = \Psi(x). \qquad (18)$$

From Eqs. (11) and (18) we now have to choose $f(x)$ and $g(x)$ so that

$$f(x) + g(x) = \Phi(x), \qquad -cf'(x) + cg'(x) = \Psi(x).$$

The second gives

$$f'(x) - g'(x) = (-1/c)\Psi(x)$$

$\Rightarrow$

$$f(x) - g(x) = (-1/c)\int_d^x \Psi(s)ds,$$

where d is a constant. Thus

$$\begin{aligned} f(x) &= \frac{1}{2}\Phi(x) - \frac{1}{2c}\int_d^x \Psi(s)ds, \\ g(x) &= \frac{1}{2}\Phi(x) + \frac{1}{2c}\int_d^x \Psi(s)ds, \end{aligned}$$

and from Eq. (11) $\Rightarrow$

$$\begin{aligned} y(x,t) &= \frac{1}{2}\{\Phi(x-ct) + \Phi(x+ct)\} \\ &+ \frac{1}{2c}\int_d^{x+ct} \Psi(s)ds - \frac{1}{2c}\int_d^{x-ct} \Psi(s)ds \end{aligned}$$

$\Rightarrow$

$$y(x,t) = \frac{1}{2}\{\Phi(x-ct) + \Phi(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(s)ds. \tag{19}$$

Example 3

Given that $\Phi(x) = a\cos(kx),\ \Psi(x) = -kca\sin(kx)$ in Eq. (18), find $y(x,t)$.

Solution

From Eq. (19),

$$\begin{aligned} y(x,t) &= \frac{a}{2}\left\{\cos(k(x-ct)) + \cos(k(x+ct))\right\} \\ &- \frac{ka}{2}\int_{x-ct}^{x+ct}\sin(ks)ds \\ &= \frac{a}{2}\left\{\cos(k(x-ct)) + \cos(k(x+ct))\right\} \\ &+ \frac{a}{2}\left[\cos(ks)\right]_{x-ct}^{x+ct} \\ &= a\cos(k(x+ct)) \end{aligned}$$

Thus the two terms in Eq. (19) combine so that the wave is purely travelling to the left.

Exercises for students:

[1] Show that Eq. (19) gives a wave travelling only to the left (i.e. $y = g(x+ct)$) if and only if $\Psi(x) = c\Phi'(x)$.
[2] What initial conditions give $y(x,t) = a\tanh(k(x-ct))$ for $-\infty < x < \infty$ and $\forall\ t \geq 0$?

1.4 Strings of finite length

- Now Eq. (17) can be written

$$(\text{since} \quad \sin A + \sin B = 2\sin\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right])$$

$$y(x,t) = a\sin(kx)\cos(kct) \qquad (20)$$

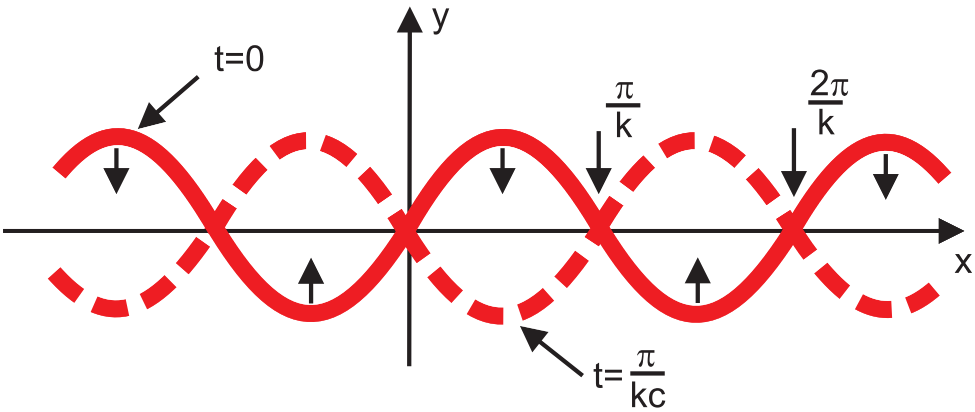

Figure 9: Standing wave

Thus y is always zero at $x = n\pi/k$.

Between $x = r_1\pi/k$ and $x = r_2\pi/k$ the string oscillates periodically in time.
Eq. (20) is an example of a standing wave, with a being the amplitude, k the wavenumber ($k > 0$), $2\pi/k$ the wavelength. The period of oscillation is $2\pi/kc$.

• Standing waves occur with a string of finite length L. Suppose the string is fixed at $x = 0$, $x = L$ (e.g., a piano wire or violin) so the solution of Eq. (5), the wave equation, must satisfy

$$y(0,t) = y(L,t) = 0. \tag{21}$$

We look for solutions of Eq. (5) of the form (separable solutions)

$$y(x,t) = X(x)T(t) \tag{22}$$

Substituting in Eq. (5) ⇒

$$c^2 X'' T = X \ddot{T}$$

⇒

$$\frac{X''}{X} = \frac{1}{c^2}\left(\frac{\ddot{T}}{T}\right).$$

The LHS depends only on x, the RHS depends only on t so the equation can be true for ∀ (x,t) only if each side is a constant. There are three cases to consider.

[1] Constant $> 0 = k^2$

$\Rightarrow$

$$X'' = k^2 X$$

$\Rightarrow$

$$X = A\cosh(kx) + B\sinh(kx).$$

From Eq. (21) $\Rightarrow A = B = 0$. Not useful.

[2] Constant=0

$\Rightarrow$

$$X'' = 0$$

$\Rightarrow$

$$X = Ax + B.$$

From Eq. (21) $\Rightarrow A = B = 0$. Not useful.

[3] Constant $< 0 = -k^2$

$\Rightarrow$

$$\begin{aligned} X'' &= -k^2 X, \\ \ddot{T} &= -k^2 c^2 T. \end{aligned} \tag{23}$$

First of Eq. (23) $\Rightarrow X = A\cos(kx) + B\sin(kx)$.

From Eq. (21):

$$\begin{aligned} y(0,t) = 0 &\;\Rightarrow\; A = 0 \Rightarrow y = B\sin(kx) \\ y(L,t) = 0 &\;\Rightarrow\; B\sin(kL) = 0. \end{aligned}$$

For useful/interesting results we cannot have $B = 0$ which implies $sin(kl) = 0 \Rightarrow kL = n\pi$ (n$= 0, 1, 2...$)

$\Rightarrow$

$$X = B_n sin(n\pi x/L)$$

and

$$\ddot{T} = -(n\pi c/L)^2 T.$$

$\Rightarrow$

$$T = \alpha\cos(n\pi ct/L) + \beta\sin(n\pi ct/L).$$

Thus a solution of Eq. (5) (wave equation) of the form Eq. (22) (separable solutions) satisfying Eq. (21) (fixed boundary) is

$$y = \sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \quad (n = 1, 2, 3...). \tag{24}$$

For each n, the solution in Eq. (24) is a periodic wave [like Eq. (20)] with period $2\pi L/n\pi c = 2L/nc$.

We often rewrite

$$\begin{matrix} \cos(n\pi ct/L) & & cos(\omega_n t) \\ & \text{as} & \\ \sin(n\pi ct/L) & & sin(\omega_n t) \end{matrix}$$

where ω_n is the angular frequency:

$$\omega_n = \frac{n\pi c}{L}. \tag{25}$$

Each of the solutions in Eq. (24) is a normal mode of vibration.

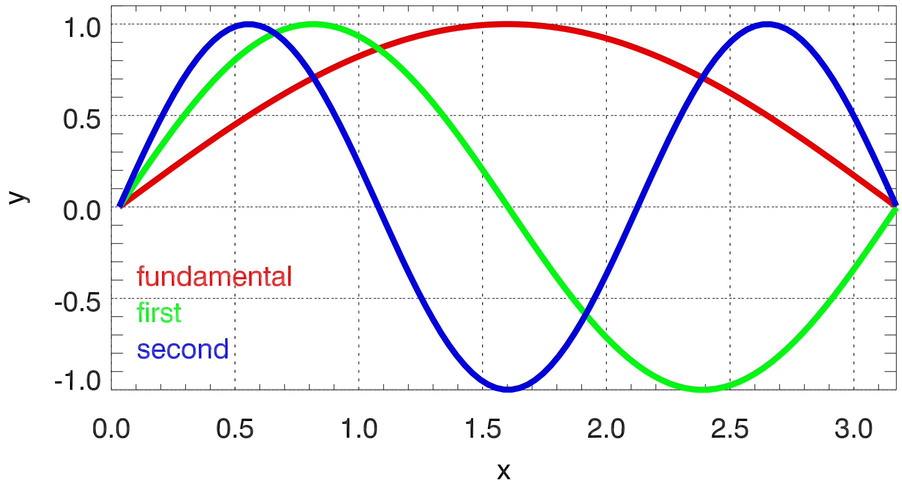

Figure 10: Standing fundamental, 1st, and 2nd harmonics

• Now Eq. (5) is a linear equation so any linear combination of the solutions in Eq. (24) is also a solution. This is the principle of superposition. Thus

$$y = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\} \tag{26}$$

is a solution of Eq. (5) satisfying Eq. (21). It is in fact the general solution of Eq. (5)-(21); the constants α_n, β_n are determined by the initial conditions (see Chapter Two).

Question: In general, is this solution periodic in time? Explain your answer.

1.5 Some technical remarks

- Consider the real part, $\Re$, of the complex quantity

$$A \exp[i(kx - \omega t)],$$

where k and ω are real but

$$A = A_r + iA_i$$

is complex. Now

$$\begin{aligned} \Re\{A \exp[i(kx - \omega t)]\} &= A_r \cos(kx - \omega t) - A_i \sin(kx - \omega t) \\ &= \sqrt{A_r^2 + A_i^2} \cos[(kx - \omega t) + \epsilon] \end{aligned}$$

where

$$\cos \epsilon = A_r / \sqrt{A_r^2 + A_i^2}, \qquad \sin \epsilon = A_i / \sqrt{A_r^2 + A_i^2}.$$

We shall consider situations in which the dependent variable, say ϕ, has the form

$$\phi = \alpha \cos[(kx - \omega t) + \epsilon]$$

(or with sin instead of cos).
Note: $\phi = \sin kx[(-\alpha \sin \epsilon) \cos \omega t + (\alpha \cos \epsilon) \sin \omega t] + \cos kx[(-\alpha \cos \epsilon) \cos \omega t + (\alpha \sin \epsilon) \sin \omega t]$, and the first term is equivalent to Eq. (24).

In linear problems it is often convenient to write (A complex; k, ω real)

$$\phi = A \exp[i(kx - \omega t)]; \tag{27}$$

we do of course really mean the real part of Eq. (27) but many problems can be solved most easily by working directly with Eq. (27) and only taking the real part right at the end.

In Eq. (27), k is again the wavenumber and ω is the angular frequency.

To satisfy the 1D wave equation Eq. (5), $\omega = kc$. The period is $2\pi/\omega$ and the frequency is $\omega/2\pi$. The frequency, measured in s^{-1} (Hz, hertz), is the number of complete oscillations that the wave makes during 1 sec at a fixed position. Finally,

$$|A| = \sqrt{A_r^2 + A_i^2}$$

is the amplitude. Eq. (27) is a periodic or harmonic wave.

2 Use of Fourier Series

2.1 Aim of Chapter

• In § 1.4, we saw that a solution of the PDE for $y(x, t)$

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad \text{with} \qquad y(0, t) = y(L, t) = 0, \qquad (1)$$

is Eq (1.24), viz.

$$y(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right]. \qquad (2)$$

Eq. (2) is in fact the general solution of (1). Our aim is to show how the constants $\{\alpha_n\}$ and $\{\beta_n\}$ can be determined, and to indicate some extensions.

• The constants $\{\alpha_n\}$ and $\{\beta_n\}$ are determined uniquely by the initial conditions, i.e. the value of $y(x, 0)$ and $\dot{y}(x, 0)$ (or, more generally, by the values of $y(x, t_0)$, $\dot{y}(x, t_0)$ for any t_0). In any particular motion, the values of $y(x, 0)$ and $\dot{y}(x, 0)$ can be chosen independently of one another.

- We note from Eq. (2) that

$$\begin{aligned} \dot{y}(x,t) &= \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ &\times \left[-\alpha_n \sin\left(\frac{n\pi ct}{L}\right) + \beta_n \cos\left(\frac{n\pi ct}{L}\right)\right]. \end{aligned} \tag{3}$$

- Thus, from Eqs. (2) and (3)

$$y(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right), \tag{4a}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \beta_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \tag{4b}$$

2.2 Worked Examples

Example 1

Find $\{\alpha_n\}$, $\{\beta_n\}$ for the case of a plucked string of length L, with its ends fixed, released from rest when the mid-point is drawn aside through a distance h. Thus

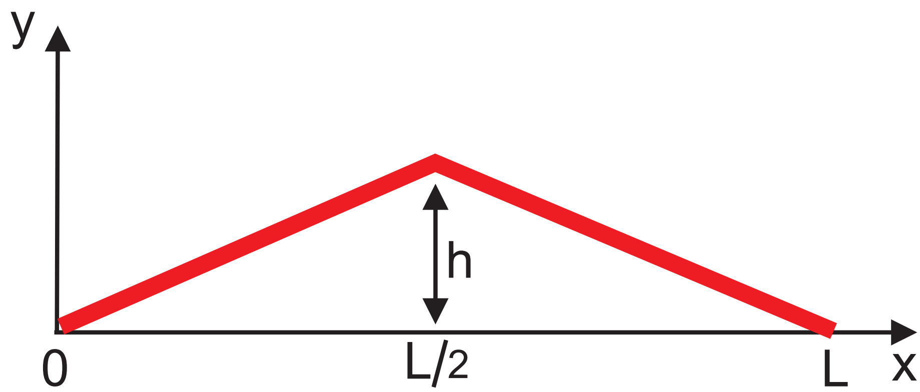

Figure 1: Plucked string with length L

$$y(x,0) = \begin{cases} \frac{2h}{L}x & (0 \le x \le \frac{1}{2}L) \\ \frac{2h}{L}(L-x) & (\frac{1}{2}L \le x \le L) \end{cases} \tag{5}$$

and

$$\dot{y}(x,0) = 0. \tag{6}$$

Solution

- Comparing Eqs. (6) and (4b), we can reconcile them by taking

$$\beta_n = 0. \tag{7}$$

It remains to reconcile Eqs. (5) and (4a). The key is to multiply Eq. (4a) by

$$\sin\left(\frac{m\pi x}{L}\right)$$

and integrate from $x = 0$ to $x = L$. Thus

$$\int_0^L y(x,0)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty}\alpha_n\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{8}$$

- Consider I_{mn}, where

$$I_{mn} = \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{9}$$

Question: How to proceed with the evaluation of I_{mn}?

Now

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

for $\forall$ A and B, so the integrand in Eq. (9) is

$$\frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) - \cos\left(\frac{(m+n)\pi x}{L}\right)\right].$$

Since m and n are positive integers, there are two possible cases.

$m \neq n$:

$$I_{mn} = \frac{l}{2\pi}\left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{(m-n)} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{(m+n)}\right]_0^L = 0$$

$m = n$:

$$I_{mn} = \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2m\pi x}{L}\right)\right] dx = \frac{L}{2}.$$

$\Rightarrow$

$$I_{mn} = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n. \end{cases} \tag{10}$$

• Hence the RHS of Eq. (8) reduced to $\frac{L}{2}\alpha_m$, and Eq. (8) can be then rewritten

$$\alpha_m = \frac{2}{L}\int_0^L y(x,0)\sin\left(\frac{m\pi x}{L}\right)dx, \tag{11}$$

where Eq. (11) is a general formula. In our particular case, use of Eq. (5) gives

$$\begin{aligned}
\alpha_m &= \frac{4h}{L^2}\int_0^{L/2} x\sin\left(\frac{m\pi x}{L}\right)dx \\
&+\frac{4h}{L^2}\int_{L/2}^{L}(L-x)\sin\left(\frac{m\pi x}{L}\right)dx, \\
&= \left\{\frac{4h}{Lm\pi}\left[-x\cos\left(\frac{m\pi x}{L}\right)\right]_0^{L/2}\right. \\
&\left.+\frac{4h}{Lm\pi}\int_0^{L/2}\cos\left(\frac{m\pi x}{L}\right)dx\right\}+ \\
&+\left\{\frac{4h}{Lm\pi}\left[-(L-x)\cos\left(\frac{m\pi x}{L}\right)\right]_{L/2}^{L}\right. \\
&\left.-\frac{4h}{Lm\pi}\int_{L/2}^{L}\cos\left(\frac{m\pi x}{L}\right)dx\right\}, \\
&= -\frac{2h}{m\pi}\cos\left(\frac{m\pi}{2}\right)+\frac{2h}{m\pi}\cos\left(\frac{m\pi}{2}\right)+\frac{4h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right) \\
&+\frac{4h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right)=\frac{8h}{m^2\pi^2}\sin\left(\frac{m\pi}{2}\right).
\end{aligned}$$

Thus

$$\alpha_m = \begin{cases} 0 & (m = 2p) \\ \frac{8h(-1)^p}{\pi^2(2p+1)^2} & (m = 2p+1). \end{cases} \tag{12}$$

• Use of Eqs. (7) and (12) upon substitution into Eq. (2) gives

$$y(x,t) = \frac{8h}{\pi^2} \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)^2} \sin\left(\frac{(2p+1)\pi x}{L}\right) \times \cos\left(\frac{(2p+1)\pi ct}{L}\right). \tag{13}$$

Example 2

Find $\{\alpha_n\}$, $\{\beta_n\}$ for the case of a string of length L, given that Eq. (1) holds and in addition $y(x,0) = 0$ and $\dot{y}(x,0) = 4Vx(L-x)/L^2$.

Solution

In this case $y(x,0) = 0 \Rightarrow$

$$\alpha_n = 0. \tag{14}$$

Then from Eq. (4b) $\Rightarrow$

$$\int_0^L \frac{4Vx(L-x)}{L^2} \sin\left(\frac{m\pi x}{L}\right) dx = \beta_m \left(\frac{m\pi c}{L}\right) \frac{L}{2},$$

using the same technique that leads from Eq. (8) to Eq. (11).

Hence

$$\begin{aligned}
\beta_m \frac{m\pi c}{2} &= \frac{4V}{L^2}\left\{\underbrace{\left[-\frac{L}{m\pi}x(L-x)\cos\left(\frac{m\pi x}{L}\right)\right]_0^L}_{=0}\right. \\
&\quad \left. + \frac{L}{m\pi}\int_0^L (L-2x)\cos\left(\frac{m\pi x}{L}\right)dx\right\}, \\
&= \frac{4V}{L^2}\left\{0 + \frac{L}{m\pi}\int_0^L (L-2x)\cos\left(\frac{m\pi x}{L}\right)dx\right\}.
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\beta_m &= \frac{8V}{m^2\pi^2 cL}\left\{\left[\frac{L}{m\pi}(L-2x)\sin\left(\frac{m\pi x}{L}\right)\right]_0^L\right. \\
&\quad \left. +\frac{2L}{m\pi}\int_0^L \sin\left(\frac{m\pi x}{L}\right)dx\right\}, \\
&= \frac{16VL}{m^4\pi^4 c}\left[\cos\left(\frac{m\pi x}{L}\right)\right]_L^0 \\
&= \frac{16VL}{m^4\pi^4 c}[1-(-1)^m].
\end{aligned}$$

Thus

$$\beta_m = \begin{cases} 0 & (m = 2p) \\ \frac{32VL}{\pi^4 c(2p+1)^4} & (m = 2p+1) \end{cases}, \tag{15}$$

and so

$$y(x,t) = \frac{32VL}{\pi^4 c} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4} \sin\left(\frac{(2p+1)\pi x}{L}\right) \times \sin\left(\frac{(2p+1)\pi ct}{L}\right). \tag{16}$$

Note: The sketches on the hand-out show how series like Eqs. (13) and (16) converge. Discontinuities, like that in the gradient of $y(x,0)$ at $x = L/2$ in Example 1, cause the coefficients in the series to decrease less rapidly with n when then there are no discontinuities. Compare the rates of fall-off with p of the coefficients in Eqs. (13) and (16). Thus Eq. (16) indicates a “purer” tone than Eq. (13).

2.3 Energy

• Consider a string occupying $0 \le x \le L$ with $y(0,t) = 0$, $y(L,t) = 0$, and consider the normal mode Eq. (1.24)

$$y = \sin\left(\frac{n\pi x}{L}\right)\left\{\alpha_n \cos\left(\frac{n\pi ct}{L}\right) + \beta_n \sin\left(\frac{n\pi ct}{L}\right)\right\}.$$

We rewrite this in the form (with $A_n \ge 0$, $0 \le \epsilon_n \le 2\pi$):

$$y = A_n \cos\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \sin\left(\frac{n\pi x}{L}\right). \qquad (17)$$

In Eq. (17) A_n is the amplitude and ϵ_n is the phase.

Note:

$$\begin{aligned} A_n \cos\{n\pi ct/L + \epsilon_n\} &= A_n \cos\epsilon_n \cos(n\pi ct/L) \\ &- A_n \sin\epsilon_n \sin(n\pi ct/L) \end{aligned}$$

so Eqs. (1.24) and (17) are the same provided

$$A_n \cos\epsilon_n = \alpha_n, \qquad A_n \sin\epsilon_n = -\beta_n.$$

$\Rightarrow$

$$A_n^2(\cos^2\epsilon_n + \sin^2\epsilon_n) = \alpha_n^2 + \beta_n^2$$

$\Rightarrow$

$$A_n = +\sqrt{\alpha_n^2 + \beta_n^2}, \qquad \tan\epsilon_n = -\beta_n/\alpha_n.$$

• By Eq. (1.7), the kinetic energy T_n associated with Eq. (17) is

$$T_n = \frac{1}{2}\rho A_n^2 \left(\frac{n\pi c}{L}\right)^2 \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx,$$

$\Rightarrow$

$$T_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \qquad (18)$$

Likewise, by Eq. (1.8), the potential energy V_n associated with Eq. (17) is

$$\begin{aligned} V_n &= \frac{1}{2} F A_n^2 \left(\frac{n\pi}{L}\right)^2 \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\} \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx, \\ &= \frac{F\pi^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \end{aligned}$$

From Eq. (1.6) $F = \rho c^2$, $\Rightarrow$

$$V_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\}. \qquad (19)$$

The total energy therefore $E_n = T_n + V_n$ is given by

$$E_n = \frac{\rho\pi^2 c^2 n^2 A_n^2}{4L} = \frac{\rho L}{4}\omega_n^2 A_n^2, \qquad \omega_n = \frac{n\pi c}{L}, \qquad (20)$$

where ω_n is the angular frequency of this normal mode.

$\Rightarrow E_n \propto A_n^2$ and $E_n \propto \omega_n^2$.

$E_n \propto A_n^2$ indicates that a much bigger proportion of the total energy is contained in the first few modes of, say, Eq. (16) than in the same number of modes of, say, Eq. (13). **Check it!**

• Now consider the general motion given by Eq. (1.24). Because, for $m \neq n$,

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = 0,$$

(the first leads to Eq. (10b) and the second is proved likewise), it follows immediately that

$$T = \sum_{n=1}^{\infty} T_n = \frac{\rho\pi^2 c^2}{4L}\sum_{n=1}^{\infty} n^2 A_n^2 \sin^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\},$$

$$V = \sum_{n=1}^{\infty} V_n = \frac{\rho\pi^2 c^2}{4L}\sum_{n=1}^{\infty} n^2 A_n^2 \cos^2\left\{\frac{n\pi ct}{L} + \epsilon_n\right\},$$

$$E = T + V = \frac{\rho\pi^2 c^2}{4L}\sum_{n=1}^{\infty} n^2 A_n^2. \tag{21}$$

• Apply to **Example 1** in § (2.2). From Eq. (12) we have

$$\begin{aligned} A_{2n} &= 0 \\ A_{2n+1} &= \frac{8h}{\pi^2(2n+1)^2}. \end{aligned}$$

Thus, from the last of Eq. (21),

$$E = \frac{\rho\pi^2 c^2}{4L}\sum_{n=0}^{\infty}\frac{64(2n+1)^2h^2}{\pi^4(2n+1)^4},$$

and thus ⇒

$$E = \frac{16\rho h^2 c^2}{\pi^2 L}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \tag{22}$$

Amazingly it turns out that we can evaluate the infinite series in Eq. (22) by using Eq. (13)! We are **given** that $y(L/2,0) = h$ (see Eq. 5). Thus, putting $x = L/2$ and $t = 0$ in Eq. (13), we find

$$\begin{aligned} h &= \frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\left[\frac{(2n+1)\pi}{2}\right] \\ &= \frac{8h}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\times(-1)^n}{(2n+1)^2}. \end{aligned}$$

Thus

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \tag{23}$$

and Eq. (22) becomes

$$E = \frac{2\rho h^2 c^2}{L}. \tag{24}$$

We can check this result from the initial conditions when $T = 0$ and

$$\begin{aligned} V &= \frac{1}{2} F \left[\int_0^{L/2} \left(\frac{2h}{L} \right)^2 dx + \int_{L/2}^{L} \left(\frac{-2h}{L} \right)^2 \right] = \frac{2Fh^2}{L} \\ &= \frac{2\rho h^2 c^2}{L}, \end{aligned}$$

using Eq. (1.8). Thus

$$E|_{t=0} = 0 + \frac{2\rho h^2 c^2}{L}.$$

• As a matter of fact, this worked example corresponds quite closely to a violin string plucked at its mid-point.

The fundamental frequency, or pitch, is

$$\frac{\pi c}{L}\frac{1}{2\pi} = \frac{c}{2L},$$

but overtones with frequencies

$$\frac{3c}{2L}, \quad \frac{5c}{2L}, \quad \dots$$

are generated. The note heard by a listener depends on the amplitudes of the overtones; the note is not pure but the (relatively) rapid fall-off of the amplitudes means that the note is purer than that of many musical instruments, particularly the piano.

If the string had been bowed at some other point than its center, the amplitude of the overtones would have been different and thus tone would have been changed.

2.4 Two (different) extensions

2.4.1 Fourier transforms

Series like Eqs. (4a) and (4b), and (perhaps!) Eqs. (13) and (16) are known as **Fourier Series** after the great French scientist and mathematician (Jean Baptiste) Joseph Fourier (1768-1830).

The methods used in this chapter are capable of extension in many different directions. The only one I want to draw attention to here is the following. We have seen Eq. (1.17) that

$$\Phi = A \exp[ik(x - ct)]$$

is a solution of the 1D wave equation for any value of the constant k. So therefore is

$$A(k) \exp[ik(x - ct)]$$

for any function $A(k)$ and, also,

$$\int_{-\infty}^{\infty} A(k) \exp[ik(x - ct)] dk.$$

This leads/is related to Fourier Transforms or Fourier Analysis.[2]

---

[2] You might enjoy looking at "Fourier Analysis" by T.W.Kövner, CUP (1988).

2.4.2 2D wave equation

Consider a membrane, e.g. the surface of a drum. Let (x, y) denote position in the membrane and $z = z(x, y, t)$ be its transverse displacement. It can be easily shown that the governing equation for z is

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right). \tag{25}$$

We can find separable solutions - see § (1.4)- of the form

$$z = X(x)Y(y)T(t).$$

But for a drum it is more natural to use polar coordinates (r, θ) with

$$x = r\cos\theta, \quad y = r\sin\theta$$

when Eq. (25) becomes (for details see notes at the end)

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right] \tag{26}$$

- As in § (1.4) we seek separable solutions of the form

$$z = R(r)\Theta(\theta)T(t),$$

but we shorten the process by looking for normal modes with

$$T \propto e^{i\omega t}.$$

$\Rightarrow$

$$\Theta\left[r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + R\frac{d^2\Theta}{d\theta^2} = -\frac{\omega^2}{c^2}r^2 R\Theta$$

$\Rightarrow$

$$\frac{1}{R}\left[r\frac{d}{dr}\left(r\frac{dR}{dr}\right)\right] + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + k^2r^2 = 0, \qquad (27)$$

where

$$k = \frac{\omega}{c} \qquad (28)$$

- Suppose

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \text{const} := -n^2$$

$\Rightarrow$

$$\Theta \propto e^{in\theta}.$$

In practice, we must have

$$\Theta(\theta) = \Theta(\theta + 2\pi) \Rightarrow n \in N^+$$

The equation for R becomes

$$r^2 R'' + rR' + (k^2 r^2 - n^2)R = 0,$$

and the change of variable $\xi = kr$ gives

$$\xi^2 \frac{\mathrm{d}^2 \mathbf{R}}{\mathrm{d}\xi^2} + \xi \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}\xi} + (\xi^2 - \mathbf{n}^2)\mathbf{R} = \mathbf{0}. \tag{29}$$

This is known as **Bessel's equation of order** n.

We now consider only the case $n = 0 \Rightarrow$ no θ variation! The only solution of Eq. (29) that is bounded at $r = 0$ is

$$R \propto J_0(\xi) = J_0(kr), \tag{30}$$

(see hand-out).

Assume, as with a drum, that the membrane is fixed at

$$r = a \Rightarrow R = 0 \text{ when } r = a$$

$\Rightarrow$

$$J_0(ka) = 0 \Rightarrow k = \frac{\lambda_m}{a}$$

where λ_m is the m-th root of $J_0(\xi)$.

- Thus

$$z = A_m J_0\left(\frac{\lambda_m r}{a}\right) e^{i\lambda_m ct/a}$$

and the **general solution**, independent of θ, is

$$\mathbf{z = \sum_{n=1}^{\infty} A_m J_0\left(\frac{\lambda_m r}{a}\right) e^{i\lambda_m ct/a}}. \tag{31}$$

Exercise

Find $\{A_m\}$ by similar methods to those in § (2.2). (Note, there is an orthogonality relationship.)

Notes on 2D cylindrical wave equation

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$\begin{aligned}
\therefore \quad z_r &= \cos\theta z_x + \sin\theta z_y \\
z_\theta &= -r\sin\theta z_x + r\cos\theta z_y
\end{aligned}$$

$$\begin{aligned}
\therefore \quad z_x &= \cos\theta z_r - \frac{\sin\theta}{r} z_\theta \\
z_y &= \sin\theta z_r + \frac{\cos\theta}{r} z_\theta
\end{aligned}$$

$$\begin{aligned}
\therefore \quad z_{xx} &= \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta z_r - \frac{\sin\theta}{r} z_\theta\right) \\
&= \cos^2\theta z_{rr} + \frac{\cos\theta\sin\theta}{r^2} z_\theta - \frac{\cos\theta\sin\theta}{r} z_{r\theta} \\
&\quad + \frac{\sin^2\theta}{r} z_r - \frac{\sin\theta\cos\theta}{r} z_{r\theta} \\
&\quad + \frac{\sin\theta\cos\theta}{r^2} z_\theta + \frac{\sin^2\theta}{r^2} z_{\theta\theta} \\
&= \cos^2\theta z_{rr} + \frac{\sin^2\theta}{r} z_r + \frac{\sin^2\theta}{r^2} z_{\theta\theta} \\
&\quad - \frac{2\cos\theta\sin\theta}{r^2} z_{r\theta} + \frac{2\cos\theta\sin\theta}{r^2} z_\theta .
\end{aligned}$$

Likewise, after algebra:

$$\begin{aligned} z_{yy} &= \sin^2\theta z_{rr} + \frac{\cos^2\theta}{r} z_r + \frac{\cos^2\theta}{r^2} z_{\theta\theta} \\ &\quad + \frac{2\cos\theta\sin\theta}{r^2} z_{r\theta} - \frac{2\cos\theta\sin\theta}{r^2} z_\theta. \end{aligned} \tag{32}$$

Therefore

$$\begin{aligned} z_{xx} + z_{yy} &= z_{rr} + \frac{1}{r} z_r + \frac{1}{r^2} z_{\theta\theta}, \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r z_r) + \frac{1}{r^2} z_{\theta\theta}. \end{aligned}$$

Notes on Convergency of Fourier series

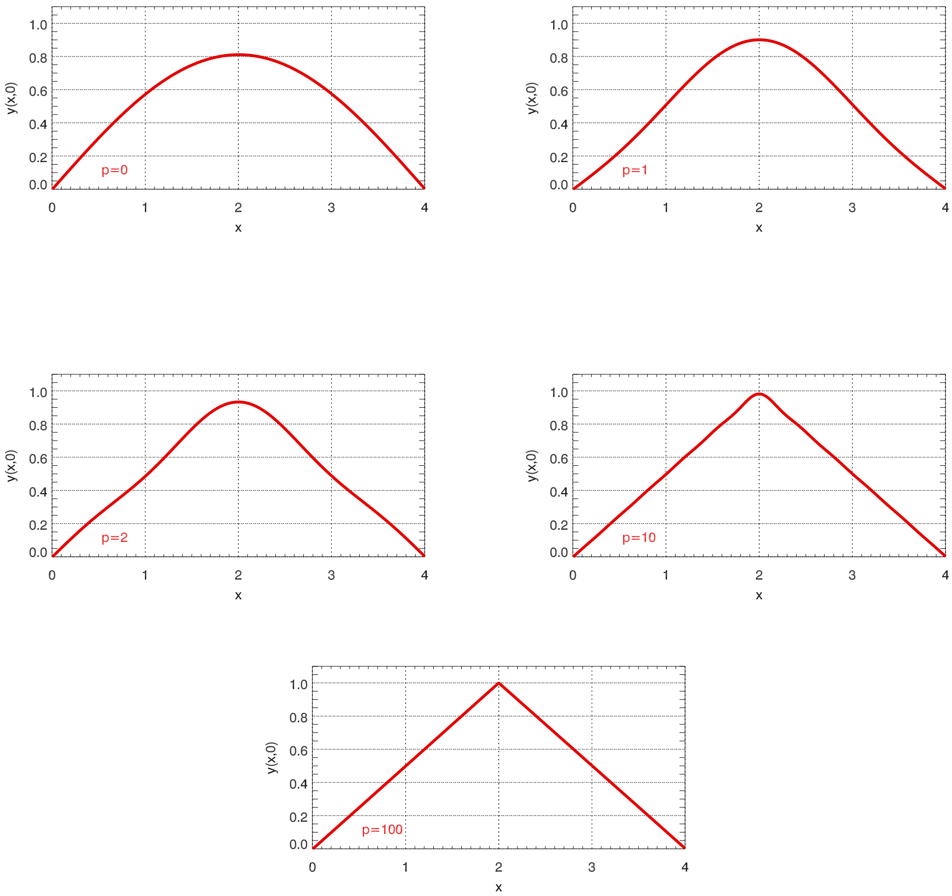

Figure 2: Convergency of Eq. (2.13) for p=0, 1, 2, 10 and 100 at t=0 (Example 1)

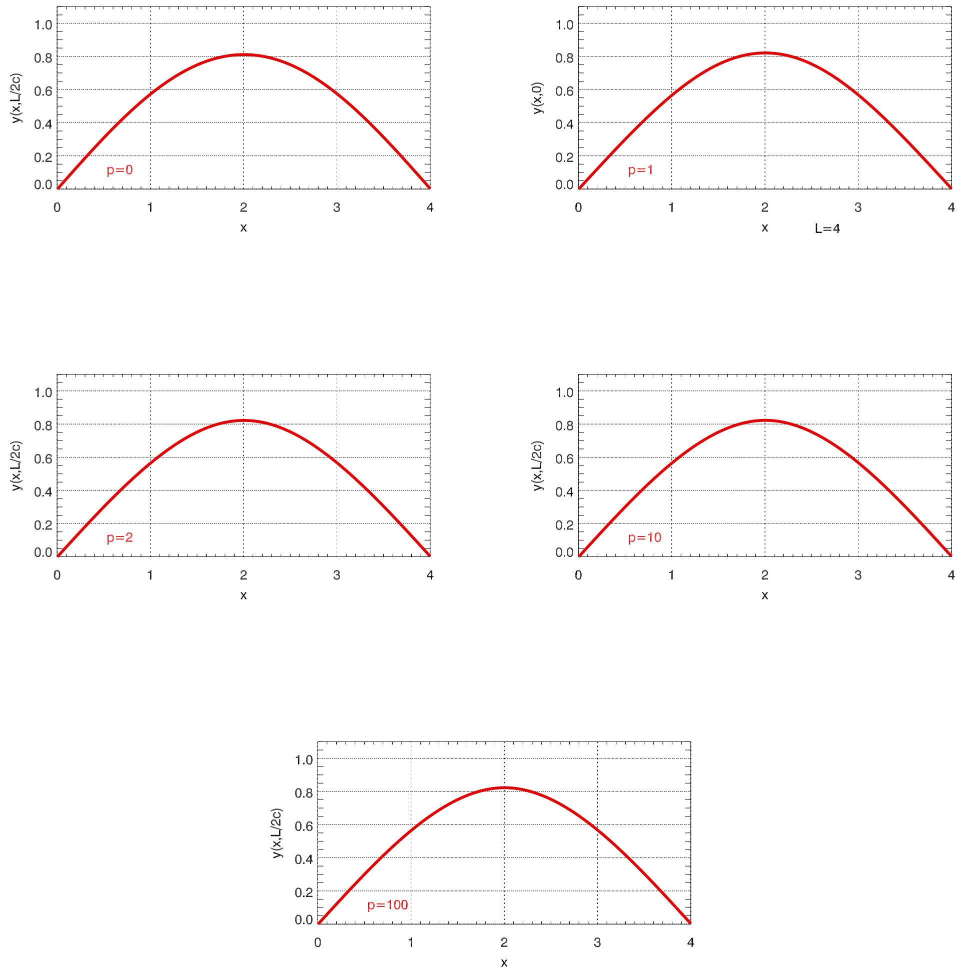

Figure 3: Convergency of Eq. (2.16) for p=0, 1, 2, 10 and 100 at $t = 0$ (Example 2)

Notes on Bessel Function J_n

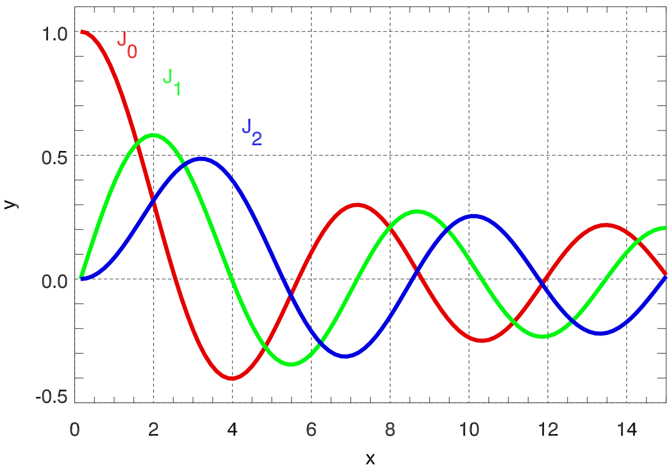

Figure 4: Bessel function J_n

- $y = J_n(x) \quad (n = 0, 1, 2, ..)$ is a solution of **Bessel's equation** of order n. This is (*see Eq. (2.29) in Notes*):

$$x^2 y'' + xy' + (x^2 - n^2)\, y = 0.$$

- $J_n(x)$ is the **Bessel function of order n** defined precisely by the infinite series

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p x^{n+2p}}{2^{n+2p}\, p!\, (n+p)!}.$$

This can be shown to satisfy Bessel's equation.

- The general solution of Bessel's equation of order n is unbounded as $x \to 0$. The most general solution that is bounded as $x \to 0$ is $y = AJ_n(x)$ where A is an arbitrary constant.

- As the sketches illustrate, $J_n(x)$ has an infinite number of zeros.

- Let α_m be the mth zero of $J_0(x)$. To good approximation

$$\alpha_1 = 2.405, \quad \alpha_2 = 5.520, \quad \alpha_3 = 8.654$$

$$\alpha_m \approx \left(m - \frac{1}{4}\right)\pi \quad \text{for large } m$$

3 Sound Waves and the Equation of Continuity

3.1 One-dimensional sound waves

• In general most phenomena associated with sound propagation can be described well by assuming that air (or the underlying fluid) has uniform density ρ_0 and uniform pressure p_0 in the undisturbed state, and that sound generation causes small changes in density ρ and pressure p, and a small velocity $\mathbf{u}$.

• We shall suppose that p and ρ are uniquely related:

$$p = p(\rho) \leftrightarrow \rho = \rho(p) \tag{1}$$

• We begin by assuming that the motion is 1D

$\Rightarrow$

$$\mathbf{u} = u(x,t)\mathbf{i}, \quad p = p(x,t), \quad \rho = \rho(x,t)$$

• Now ρ and u are not independent since mass must be conserved.

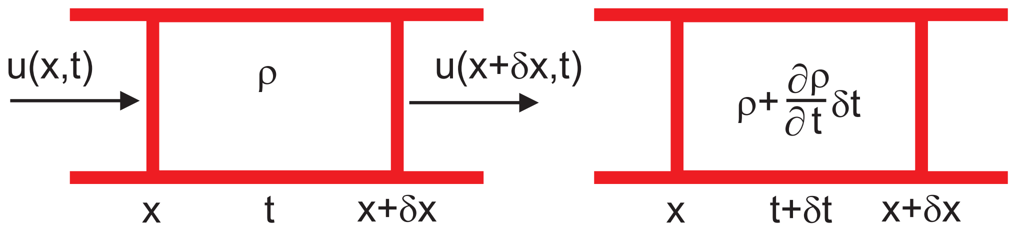

Figure 1: Conservation of mass

Consider a small tube of length δx and cross-sectional area A.

At time t the mass of fluid in the tube is

$$\rho(x,t)\delta x A$$

and at time $t + \delta t$ it is

$$[\rho(x,t) + \delta\rho]\delta x A = \rho(x,t)\delta x A + \frac{\partial \rho(x,t)}{\partial t}\delta x \delta t A.$$

The mass has increased by an amount

$$\frac{\partial \rho}{\partial t}\delta x \delta t A.$$

This is due to mass flowing into the tube: in time δt this is equal to

$$\left\{ \rho u|_{(x,t)} - \rho u|_{(x+\delta x,t)} \right\} \delta t A = -\frac{\partial}{\partial x}(\rho u)\delta x \delta t A$$
$$= -\rho \frac{\partial u}{\partial x}\delta x \delta t A - u\frac{\partial \rho}{\partial x}\delta x \delta t A.$$

But u and $\partial \rho / \partial x$ are small (we linearize) $\Rightarrow$ this is equal to

$$= -\rho_0 \frac{\partial u}{\partial x}\delta x \delta t A$$

to highest order. Thus

$$\frac{\partial \rho}{\partial t} = -\rho_0 \frac{\partial u}{\partial x} \tag{2}$$

• Now we apply N2 to the fluid in the small tube. From Fig. (2) the force on the fluid is

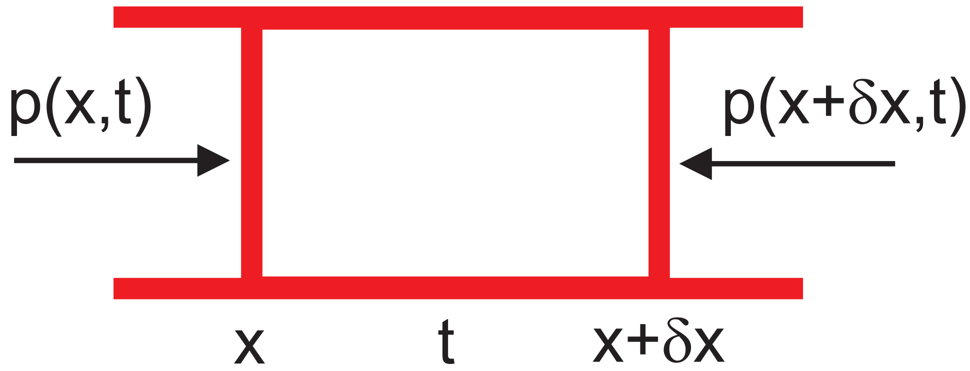

Figure 2: Forces acting on a fluid element with length δx

$$[p(x,t)A - p(x+\delta x,t)A] = -\frac{\partial p}{\partial x}\delta x A,$$

neglecting gravity and this is equal to

$$\rho \delta x A \frac{\partial u}{\partial t} \approx \rho_0 \frac{\partial u}{\partial t} \delta x A$$

by N2. Thus

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}. \tag{3}$$

- Now, by Eq. (1),

$$\begin{aligned}\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x} \\ &= \left(\frac{dp}{d\rho}\right)_{\rho=\rho_0}\frac{\partial \rho}{\partial x} \\ &:= c^2\frac{\partial \rho}{\partial x},\end{aligned}$$

where

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right). \tag{4}$$

Thus, from Eq. (2),

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho_0\frac{\partial^2 u}{\partial x \partial t} = -\rho_0\frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}c^2\frac{\partial \rho}{\partial x}\right),$$

i.e.

$$\frac{\partial^2 \rho}{\partial t^2} = c^2\frac{\partial^2 \rho}{\partial x^2}.$$

Likewise, the same equation holds for u (and p). Thus

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad \& \quad \frac{\partial^2 \rho}{\partial t^2} = c^2\frac{\partial^2 \rho}{\partial x^2} \tag{5}$$

• For sound in gases, the appropriate relationship Eq. (1) between p and ρ is the adiabatic law*

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \Rightarrow \frac{\partial p}{\partial \rho} = \frac{\gamma p_0}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}.$$

(*NB No heat exchange ⇒ change occurs quickly. Newton derived Eq. (5), but assumed that p and ρ were related by the isothermal law

$$\frac{p}{\rho} = \frac{p_0}{\rho_0}$$

Boyle's Law. ⇒

$$c = \left(\frac{p_0}{\rho_0}\right)^{1/2} \approx 280\,\text{m s}^{-1}$$

Correct expression due to Laplace.)

By Eq. (4), the velocity of sound in air is

$$c = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} \approx 330\,\text{m s}^{-1} \qquad (6)$$

at standard temperature and pressure. (T=288 K; $\gamma = 1.4$, $p_0 \approx 1.013 \times 10^5\ \text{Nm}^{-2}$, $\rho_0 \approx 1.293\ \text{kgm}^{-3}$). Eq. (6) agrees well with experiments.

The same analysis applies to water where experiments show that

$$(p - p_0) = \kappa(\rho - \rho_0)/\rho_0$$

$\Rightarrow$

$$\frac{dp}{d\rho} = \frac{\kappa}{\rho_0}$$

$\Rightarrow$

$$c \approx 1.430 \times 10^3 \text{ m s}^{-1}$$

($\kappa \approx 2.045 \times 10^9$ kgm^{-1}s^{-2}, $\rho_0 \approx 10^3$ kg m^{-3})

- One dimensional sound waves can be treated by the same mathematical methods as (one-dimensional) waves on strings. Note however that in strings the motion is transverse; in sound waves it is longitudinal.

3.2 The Equation of continuity of a fluid

• In reality, sound propagates in 3D (although the 1D result can be applied approximately to e.g. pipes). We now assume the fluid velocity $\mathbf{u}$ is still small but with three non-zero components:

$$\mathbf{u} = u(\mathbf{x}, t)\mathbf{i} + v(\mathbf{x}, t)\mathbf{j} + w(\mathbf{x}, t)\mathbf{k}. \tag{7}$$

We must again ensure that mass is conserved - the resulting equation, viz. Eq. (8), is known as the equation of continuity.

• Consider a small cuboid with sides parallel to the axes, and of lengths $\delta x, \delta y, \delta z$.

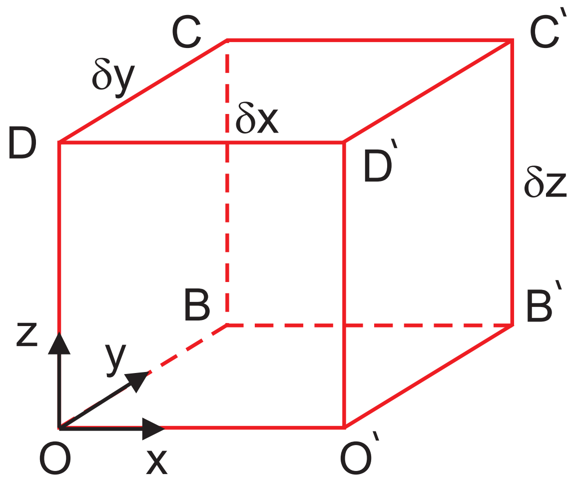

Figure 3: Three-dimensional fluid element $\delta x \delta y \delta z$

As above, the increase in the mass within this cuboid between times t and δt is

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z \delta t.$$

This is equal to the mass flowing into the cuboid, which is equal to

$$\begin{aligned}
& \left\{\rho u|_{OBCD} - \rho u|_{O'B'C'D'}\right\} \delta y \delta z \delta t \\
+ & \left\{\rho v|_{OO'D'D} - \rho v|_{BB'C'C}\right\} \delta z \delta x \delta t \\
+ & \left\{\rho w|_{OO'B'B} - \rho w|_{DD'C'C}\right\} \delta x \delta y \delta t \\
\approx & -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] \delta x \delta y \delta z \delta t.
\end{aligned}$$

Thus

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{8}$$

Eq. (8) is exact since no assumption of smallness has yet been made. This is the equation of continuity for any fluid.

3.3 Three-dimensional sound waves

- When $\rho - \rho_0$ and u, v, w are small, Eq. (8) becomes

$$\frac{\partial \rho}{\partial t}+\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\underbrace{(\rho-\rho_0)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}_{\text{2nd order}}$$

$$\underbrace{+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}}_{\text{2nd order}}=0.$$

Thus

$$\frac{\partial \rho}{\partial t}=-\rho_0\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \tag{9}$$

replaces Eq. (2) for sound waves.

- N2 gives three equations like Eq. (3)[3] $\Rightarrow$:

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x},\quad \frac{\partial v}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y},\quad \frac{\partial w}{\partial t}=-\frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{10}$$

or, using Eq. (4):

$$\frac{\partial u}{\partial t}=-\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial x},\quad \frac{\partial v}{\partial t}=-\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial y},\quad \frac{\partial w}{\partial t}=-\frac{c^2}{\rho_0}\frac{\partial \rho}{\partial z} \tag{11}$$

[3] The LHS of Eq. (10) - and Eq. (3) earlier - involve an assumption of u, v, w being small. See § (4.1)

Thus, from Eq. (9)

$$\begin{aligned}\frac{\partial^2\rho}{\partial t^2} &= -\rho_0\left(\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 v}{\partial y\partial t}+\frac{\partial^2 w}{\partial z\partial t}\right)\\ &= -\rho_0\left\{\frac{\partial}{\partial x}\left(-\frac{c^2}{\rho_0}\frac{\partial\rho}{\partial x}\right)\right.\\ &\qquad\left.+\frac{\partial}{\partial y}\left(-\frac{c^2}{\rho_0}\frac{\partial\rho}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{c^2}{\rho_0}\frac{\partial\rho}{\partial z}\right)\right\}.\end{aligned}$$

Hence

$$\frac{\partial^2\rho}{\partial t^2} = c^2\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}+\frac{\partial^2\rho}{\partial z^2}\right). \tag{12}$$

This is the 3D wave equation with speed c.

- In most circumstances $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad w = \frac{\partial\phi}{\partial z}, \tag{13}$$

and ϕ is the velocity potential. It is shown in S3 Q3 that ϕ also satisfies the same eqn. (so therefore do u, v, w):

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right). \tag{14}$$

4 Surface Waves on Liquids

4.1 Introduction

• We consider waves on the surface of liquids, e.g. waves on the sea or a lake or a river. These can be generated by the wind, by a moving boat and in many other ways. One key factor is that if the surface is displaced from its equilibrium position $z = 0$ to $z = \eta(x, y, t)$, gravity will tend to restore the surface to its equilibrium position.

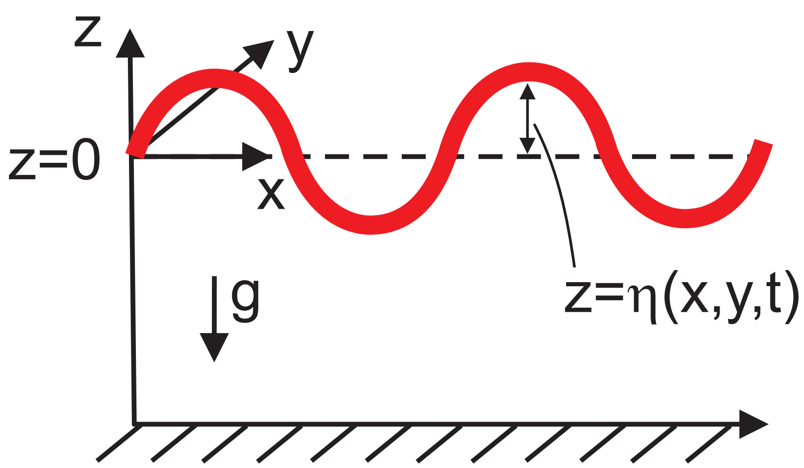

Figure 4: Surface waves

• In practice we can assume that the disturbance is small (i.e. the amplitude, e.g. $\sup|\eta|$, is much less than the wavelength) $\Rightarrow$ linear theory.

• A further simplification is that for most liquids the equation of continuity Eq. (3.8) can be considerably simplified because liquids are difficult to compress
$\Rightarrow$ volume of small piece of liquid is unchanged as it moves
$\Rightarrow$ density is unchanged (since mass = density $\times$ volume and mass is conserved).

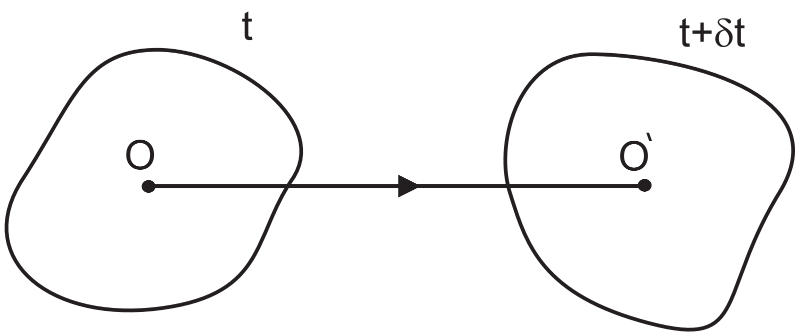

Figure 5: Incompressible liquids. $\vec{OO'} = \mathbf{u}(\mathbf{x}, t)\delta t$

Consider a small volume of liquid of density ρ. Suppose it is at $\mathbf{x}$ at time t ; in a small interval of time δt,

the volume will have moved from

$$\mathbf{x} \quad \rightarrow \quad \mathbf{x} + \mathbf{u}(\mathbf{x}, t)\delta t,$$

so the density will have changed from

$$\rho(\mathbf{x}, t) \quad \rightarrow \quad \rho(\mathbf{x} + \mathbf{u}\delta t, t + \delta t).$$

By hypothesis, these are the same. But...

$$\begin{aligned}\rho(\mathbf{x}+\mathbf{u}\delta t, t+\delta t) &= \rho(x+u\delta t, y+v\delta t, z+w\delta t, t+\delta t)\\ &\approx \rho(x,y,z,t)\\ &\quad +(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}+\frac{\partial\rho}{\partial t})\delta t.\end{aligned}$$

Thus

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}=0 \qquad (1)$$

where D/Dt is the operator defined by

$$\frac{D}{Dt}=\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right) \qquad (2)$$

D/Dt applied to any function of $(\mathbf{x}, t)$ measures rate of change when moving with the liquid (or fluid). From Eq. (3.8) we have

$$\frac{\partial\rho}{\partial t}+\left(u\frac{\partial\rho}{\partial x}+v\frac{\partial\rho}{\partial y}+w\frac{\partial\rho}{\partial z}\right)+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0,$$

so Eq. (1) $\Rightarrow$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

which is the equation of continuity for an incompressible fluid or liquid.

• The reasoning applied to $\rho(\mathbf{x}, t)$ above can also be applied to $\mathbf{u}(\mathbf{x}, t)$ (i.e. velocity).

The rate of change of the velocity of the piece of fluid, i.e. its acceleration, is

$$\begin{aligned} \frac{D}{Dt}(u, v, w) &= \frac{\partial}{\partial t}(u, v, w) \\ &\quad + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right)(u, v, w). \end{aligned}$$

But our assumption that the disturbance is small $\Rightarrow$ second term is small $\Rightarrow$

$$\text{acceleration} \approx \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t} \right).$$

(This has already been used in Eqs (3.3) and (3.10).)

4.2 The governing equations for 1D water waves

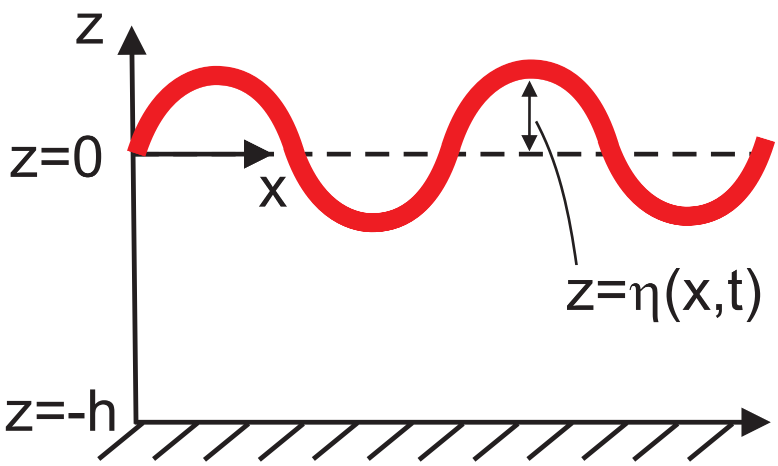

Figure 6: 1-dimensional free surface

• We consider only cases where the disturbance of the free surface is independent of $y \Rightarrow z = \eta(x,t)$ is the disturbance. We therefore assume that

$$\mathbf{u} = u(x,z,t)\mathbf{i} + w(x,z,t)\mathbf{k} \tag{4}$$

Then Eq. (3) $\Rightarrow$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{5}$$

Recalling N2 $\Rightarrow$

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \quad \rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - \rho g \tag{6}$$

where ρ can be regarded as constant*, and the term ρg represents the weight $\rho g\,\delta V$ acting vertically downwards.

*This is an extension of Eq. (1) - we assume ρ is an absolute constant, independent of both $\mathbf{x}$ and t. In the ocean, ρ does vary (slightly) with height, but not enough to affect the analysis of surface waves.

- As in § (3.3), it can be shown that, in most circumstances, there is a velocity potential, ϕ such that Eq. (3.13) holds[4]. In the present case $\phi = \phi(x, z, t)$ and

$$u = \frac{\partial \phi}{\partial x}, \quad w = \frac{\partial \phi}{\partial z} \tag{7}$$

[4]To show this is beyond the scope of this course. In brief we require the effects of viscosity to be small.

Then Eq. (5) becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{8}$$

This is the 2D form of Laplace's equation and is the PDE that must be solved. NB Surface waves are not governed by the wave equation!

• The special features of surface waves arise because of the boundary conditions. There will be three in the problems we consider:

1. $w = 0$ on $z = -h$, where h is constant (see Fig 6).
2. the vertical velocity given by $\frac{\partial \phi}{\partial z}$ at the free surface must equal the vertical velocity given by $z = \eta(x, t)$.
3. the pressure at the free surface must be continuous and since the density of air is much less than that of water, we can assume the air pressure is constant p_0.

• (1): $\Rightarrow$

$$\frac{\partial \phi}{\partial z} = 0, \quad z = -h \tag{9}$$

- (2):

$$\frac{D}{Dt}\{z - \eta(x,t)\} = 0 \quad \text{at} \quad z = \eta$$

$\Rightarrow$

$$w - \frac{\partial \eta}{\partial t} - \underbrace{u\frac{\partial \eta}{\partial x}}_{\approx \text{small}} = 0 \quad \text{at} \quad z = \eta.$$

Since we are linearising, this condition can be applied at $z = 0 \Rightarrow$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0. \tag{10}$$

- (3) : Eq. $(6) \Rightarrow$

$$\frac{\partial}{\partial x}\left\{\frac{p - p_0}{\rho} + \dot{\phi}\right\} = \frac{\partial}{\partial z}\left\{\frac{p - p_0}{\rho} + \dot{\phi} + gz\right\} = 0$$

$\Rightarrow$

$$\frac{p - p_0}{\rho} + \frac{\partial \phi}{\partial t} + gz \quad \text{depends only on} \quad t.$$

However we can incorporate this function of t by adding it to ϕ.

This has no effect on $\mathbf{u}$ by Eq. (7). Since $p = p_0$ at $z = \eta$, and we are linearising, we can apply this condition at $z = 0$ as far as ϕ is concerned. Thus from (3) $\Rightarrow$

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at} \quad z = 0. \tag{11}$$

4.3 Monochromatic surface waves

• Monochromatic $\Rightarrow$ single wave number k, single (angular) frequency ω. We assume the free surface is given by

$$\begin{aligned} \eta = \eta_0 \sin(kx - \omega t) \; &= \; \eta_0 \sin k(x - ct) \\ \omega \; &= \; kc \end{aligned} \tag{12}$$

Eq. (12b) has already been used several times, e.g. Eq. (2.28). We could also work with the complex form Eq. (1.27), viz

$$\eta = \eta_0^* \mathrm{e}^{i(kx - \omega t)}.$$

In order to satisfy Eqs. (10) and (11) we must have

$$\phi = f(z) \cos(kx - \omega t) \tag{13}$$

where Eq. (8) $\Rightarrow$

$$f'' = k^2 f. \tag{14}$$

In view of the BC (9), it is convenient to write the GS of Eq. (14) in the form

$$f = A\cosh k(z+h) + B\sinh k(z+h),$$

when from Eq. (9) $\Rightarrow B = 0$, so

$$\phi = A\cosh k(z+h)\cos(kx - \omega t) \tag{15}$$

[or : GS of Eq. (14) is, using exp functions,

$$f = \gamma \mathrm{e}^{kz} + \delta \mathrm{e}^{-kz}.$$

Eq. (9) $\Rightarrow$

$$k\gamma \mathrm{e}^{-kh} - k\delta \mathrm{e}^{kh} = 0 \quad \Rightarrow \quad \delta = \gamma \mathrm{e}^{-2kh}.$$

Thus

$$\begin{aligned} f &= \gamma \mathrm{e}^{kz} + \gamma \mathrm{e}^{-2kh}\mathrm{e}^{-kz} = \gamma \mathrm{e}^{-kh}\mathrm{e}^{k(z+h)} + \gamma \mathrm{e}^{-kh}\mathrm{e}^{-k(z+h)} \\ &= 2\gamma \mathrm{e}^{-kh}\cosh k(z+h) = A\cosh k(z+h) \end{aligned}$$

with

$$A = 2\gamma \mathrm{e}^{-kh}.]$$

- There remain Eqs. (10) and (11).

From Eq. (10) $\Rightarrow$

$$-\omega\eta_0\cos(kx-\omega t)=kA\sinh kh\cos(kx-\omega t)$$

$\Rightarrow$

$$-\omega\eta_0=kA\sinh kh \qquad \text{(A)}$$

From Eq. (11) $\Rightarrow$

$$\omega A\cosh kh\sin(kx-\omega t)+g\eta_0\sin(kx-\omega t)=0$$

$\Rightarrow$

$$-g\eta_0=\omega A\cosh kh \qquad \text{(B)}$$

Then (A)/(B) $\Rightarrow$

$$\frac{\omega}{g}=\frac{k}{\omega}\tanh kh$$

$\Rightarrow$

$$\omega^2 = gk \tanh kh, \qquad c^2 = \frac{g}{k} \tanh kh \tag{16}$$

and

$$\phi = -\frac{g\eta_0}{\omega} \frac{\cosh k(z+h)}{\cosh kh} \cos(kx - \omega t). \tag{17}$$

Thus waves with different wavelengths travel at different speeds c, where

$$c = \frac{\omega}{k} \tag{18}$$

is the phase velocity (speed). This phenomenon is known as dispersion.

We note two special cases:

$$\text{Deep water} \quad h \to \infty, \quad \omega^2 = gk, \quad c^2 = \frac{g}{k} \tag{19a}$$

$$\text{Shallow water} \quad kh \ll 1, \quad \omega^2 \approx gk^2 h \quad c^2 = gh \tag{19b}$$

NB: Shallow water waves are not dispersive. This is a progressive wave, but standing waves can be dealt with similarly - see S4 Q3.

4.4 Energy

- The PE relative to the the undisturbed position is

$$\left(\int_0^\eta \rho g z dz\right)\delta A = \frac{1}{2}\rho g \eta^2 \delta A.$$

Thus the PE in a wavelength per unit width in the direction of $0y$ is V_*, where

$$V_* = \frac{1}{2}\rho g \eta_o^2 \int_0^{2\pi/k} \sin^2 k(x-ct)dx$$

using Eq. (12). This is

$$\frac{1}{2}\rho g \eta_0^2 \cdot \frac{\pi}{k} = \frac{1}{4}\rho g \eta_0^2 \lambda,$$

where $\lambda = \frac{2\pi}{k}$ is the wavelength. Thus the potential energy density per unit area of water surface is $V_\rho = V_*/\lambda$.

$$V_\rho = \frac{1}{4}\rho g \eta_0^2 \tag{20a}$$

• Likewise the KE in a wavelength per unit width in the direction of $0y$ is T_*, where

$$T_* = \frac{1}{2}\rho \int_{-h}^{0} dz \int_{0}^{2\pi/k} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right] dx$$

$$\left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{g^2 \eta_0^2 k^2}{\omega^2 \cosh^2 kh} \cosh^2 k(z+h) \sin^2 k(x-ct)$$

$$\left(\frac{\partial \phi}{\partial z}\right)^2 = \frac{g^2 \eta_0^2 k^2}{\omega^2 \cosh^2 kh} \sinh^2 k(z+h) \cos^2 k(x-ct)$$

Since (as with V_*)

$$\int_0^{2\pi/k} \sin^2 k(x-ct)dx = \int_0^{2\pi/k} \cos^2 k(x-ct)dx = \frac{\pi}{k},$$

we find

$$T_* = \frac{\pi}{2}\frac{\rho g^2 \eta_0^2 k}{\omega^2 \cosh^2 kh}\int_{-h}^{0} \cosh 2k(z+h)dz,$$

(since $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$). Thus

$$T_* = \frac{\pi}{4}\frac{\rho g^2 \eta_0^2}{\omega^2 \cosh^2 kh} \cdot \sinh 2kh = \frac{\pi}{2}\frac{\rho g^2 \eta_0^2}{\omega^2}\tanh kh$$

(since $\sinh 2\theta = 2\sinh\theta\cosh\theta$).

$\Rightarrow$ from Eq. (16),

$$T_* = \frac{1}{2}\rho g \eta_0^2 \frac{\pi}{k} = \frac{1}{4}\rho g \eta_0^2 \lambda = V_*.$$

$\Rightarrow$ the kinetic energy density per unit area of the water surface is $T_\rho = T_*/\lambda$

$$T_\rho = \frac{1}{4}\rho g \eta_0^2. \tag{20b}$$

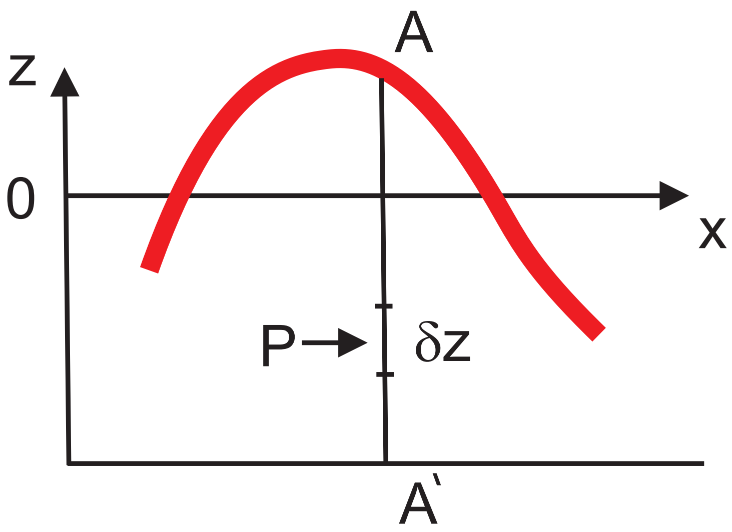

Figure 7: 1-dimensional distorted free surface

• We now calculate the rate at which work is being done by the fluid on the left of AA' on the fluid on the right. The force per unit width in the direction of $0y$ is $p\,\delta z$ so its rate of working P_* (for power) per unit width is given by

$$P_* = \int_{-h}^{0} pu\,dz = \int_{-h}^{0} \left(p_0 - \rho\frac{\partial\phi}{\partial t} - gz\right)\frac{\partial\phi}{\partial x}dz,$$

since $p = p_0 - \rho\frac{\partial\phi}{\partial t} - gz$ from derivation of Eq. (11). It is sufficient for our purposes to calculate the mean of P_* over one period. Since the mean of $\sin k(x-ct)$ is 0, and the mean of $\sin^2 k(x-ct) = \frac{1}{2}$, we let P be the mean of P_* and find:

$$P = \frac{\rho g^2 \eta_0^2 k}{2\omega \cosh^2 kh} \int_{-h}^{0} \cosh^2 k(z+h) dz$$

after some algebra (exercise for student). Since $c = \omega/k$ and $\cosh^2\theta = \frac{1}{2}(1+\cosh 2\theta)$, we find

$$\begin{aligned} P &= \frac{\rho g^2 \eta_0^2}{4c\cosh^2 kh}\left[h + \frac{\sinh 2kh}{2k}\right] \\ &= \frac{\rho g^2 \eta_0^2}{8kc}\left[1 + \frac{2kh}{\sinh 2kh}\right] 2\tanh kh \end{aligned}$$

since $\sinh 2\theta = 2\sinh\theta\cosh\theta$. Thus, using Eq. (16),

$$P = \frac{1}{4}\rho g \eta_0^2 c \left[1 + \frac{2kh}{\sinh 2kh}\right]. \tag{21}$$

• There is an interesting and important consequence of Eqs. (20a), (20b) and (21) which can be extended to many sorts of waves leading to the concept of group velocity.

As a consequence of the passage of the waves, energy is being transmitted from left to right with a (mean) speed U to be determined.

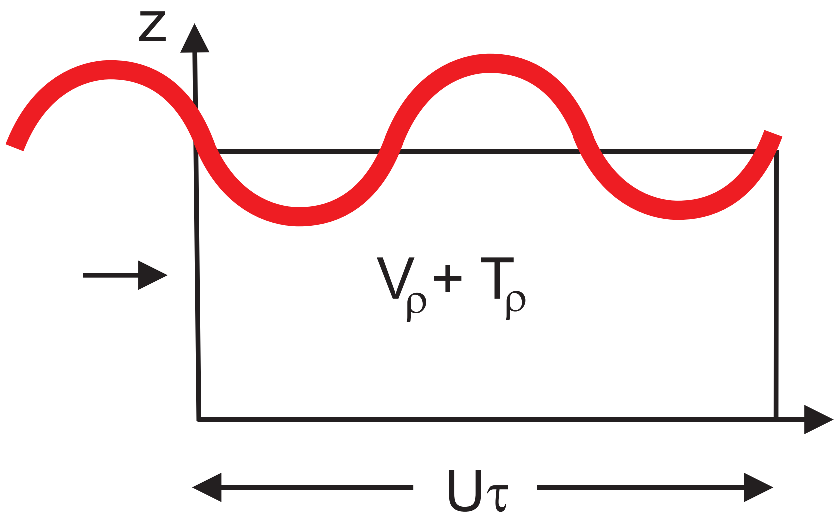

Figure 8: Concept of group velocity

In a time τ, this results in new energy per unit width equal to $(V_\rho + T_\rho)U\tau$, and this must be equal to $P\tau$, the work done. $\Rightarrow$

$$U = P/(V_\rho + T_\rho) = P/(\frac{1}{2}\rho g\eta_0^2),$$

i.e.

$$U = c_g = \frac{1}{2}c\left[1 + \frac{2kh}{\sinh 2kh}\right] \tag{22}$$

where c_g is known as the group velocity for reasons that will be discussed later.

- From the first of Eq. (16), we have

$$2\omega\frac{\mathrm{d}\omega}{\mathrm{d}k} = g\tanh kh + \frac{gkh}{\cosh^2 kh}$$

(since $\frac{\mathrm{d}}{\mathrm{d}\theta}(\tanh\theta) = \mathrm{sech}\,^2\theta = \frac{1}{\cosh^2\theta}$). Thus

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}k} &= \frac{g\tanh kh}{2\omega}\left[1 + \frac{kh}{\tanh kh\cosh^2 kh}\right]\\ &= \frac{kc^2}{2\omega}\left[1 + \frac{2kh}{\sinh 2kh}\right],\end{aligned}$$

i.e. see Eqs. (18) and (22)

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \qquad (23)$$

Eq. (23) is the general definition of group velocity.
[Note that $\omega = kc$ so that when c is independent of k, as for waves on strings and sound waves, i.e. when the waves are non-dispersive, Eq. (23) gives

$$c_g = c \qquad (24)$$

i.e. the group velocity c_g is equal to c, the phase velocity.]

• Finally, we record the results for the two special cases considered in Eqs. (19a)-(19b)

Deep water $h \to \infty \Rightarrow$

$$\omega^2 = gk, \quad c^2 = \frac{g}{k}, \quad c_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}c \tag{25a}$$

Shallow water $kh \ll 1 \Rightarrow$

$$\omega^2 \approx gk^2h, \quad c^2 = gh, \quad c_g = \sqrt{gh} = c \tag{25b}$$

4.5 Group velocity

• Energy is (often and at an average over time) transported at the group velocity c_g. This applies to many sorts of wave.

• There are other important properties of c_g. Consider the superposition of two waves like Eq. (12) in the case when the amplitudes are equal but the waves numbers and frequencies are slightly different. We have

$$\begin{aligned}\eta &= \eta_0 \sin(kx - \omega t) + \eta_0 \sin\left[(k + \delta k)x - (\omega + \delta\omega)t\right] \\ &= 2\eta_0 \sin\left[(k + \frac{1}{2}\delta k)x - (\omega + \frac{1}{2}\delta\omega)t\right] \\ &\qquad \times \cos\left[\frac{1}{2}\delta k(x - \frac{\delta\omega}{\delta k}t)\right]\end{aligned}$$

$\Rightarrow$

$$\eta \approx 2\eta_0 \cos\left[\frac{1}{2}\delta k(x - c_g t)\right] \sin[kx - \omega t] \qquad \left(c_g \approx \frac{\delta\omega}{\delta k}\right) \tag{26}$$

The combined displacement can be thought of as the original wave but with an amplitude that gradually changes between $\pm 2\eta_0$ over a distance $\pi/(\frac{1}{2}\delta k) = 2\pi/(\delta k)$.

The surface will be a series of groups of waves, separated by essentially smooth water where

$$\cos[\frac{1}{2}\delta k(x - c_g t)] \approx 0.$$

The groups are travelling at speed c_g, whereas the individual waves within each group are travelling at speed c. See top sketch in handout.
NB In passing, suppose η is density or velocity potential in sound waves, where $c_g = c$. Then Eq. (26) becomes

$$\eta \approx 2\eta_0 \cos\left[\frac{1}{2}(\delta k x - \delta\omega t)\right] \sin\left[kx - \omega t\right],$$

so that the wave has a fluctuating intensity known as beats ; the beat frequency is $\delta\omega$. This phenomenon can be used to determine unknown frequencies, by determining the beat frequency between a standard tuning fork and the unknown.

• We can develop the above analysis to consider a wave packet. As noted in § (2.5i), we can generalise to consider the disturbance $\eta(x,t)$, where

$$\eta(x,t) = \int_{-\infty}^{\infty} A(k)\mathrm{e}^{i(kx-\omega t)}dk,$$

and the real part of this is eventually to be taken. Here we shall consider the special case when

$$A(k) = A\mathrm{e}^{-d^2(k-k_0)^2},$$

where A, d, k_0 are constants. This gives the Gaussian wave packet

$$\eta(x,t) = A\int_{-\infty}^{\infty} \mathrm{e}^{-d^2(k-k_0)^2}\mathrm{e}^{i(kx-\omega t)}dk, \qquad (27)$$

The dominant contribution comes from values of k near k_0 because of the nature of $\mathrm{e}^{-d^2(k-k_0)^2}$. We write

$$\omega = \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}k}(k-k_0) + \ldots \quad = \quad \omega_0 + c_g(k-k_0) + \ldots$$

and neglect terms of higher order to obtain

$$\eta(x,t) = A\mathrm{e}^{i(k_0x-\omega_0 t)} \int_{-\infty}^{\infty} \mathrm{e}^{-d^2(k-k_0)^2+i(k-k_0)(x-c_g t)} dk$$

Substitute $d(k-k_0) = \xi$ to obtain (after some algebra):

$$\eta(x,t) = A\mathrm{e}^{i(k_0x-\omega_0 t} \int_{-\infty}^{\infty} \mathrm{e}^{-\{\xi-\frac{i}{2d}(x-c_g t)\}^2} \mathrm{e}^{-\frac{(x-c_g t)^2}{4d^2}} \frac{d\xi}{d}$$

Now substitute $\zeta = \xi - \frac{i}{2d}(x - c_g t)$ to get

$$\eta(x,t) = \frac{A}{d}\mathrm{e}^{-(x-c_g t)^2/(4d^2)} \mathrm{e}^{i(k_0x-\omega_0 t)} \int_{-\infty}^{\infty} \mathrm{e}^{-\zeta^2} d\zeta$$

Now

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\zeta^2} d\zeta = \sqrt{\pi} \qquad (28)$$

Proof of Eq. (28) is via a clever trick!

Proof:

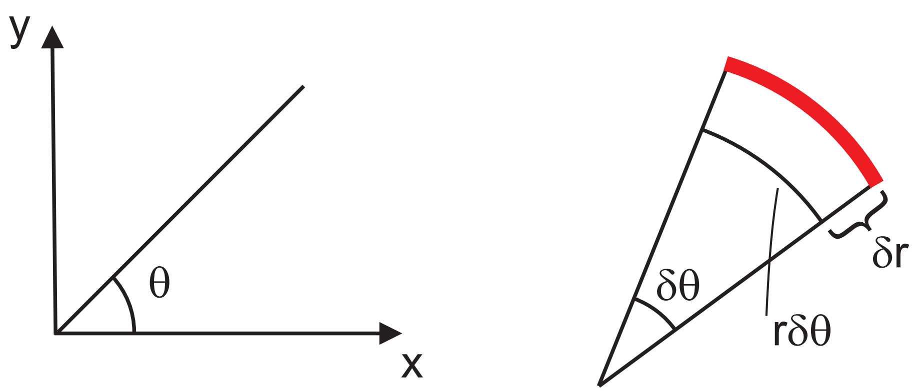

Figure 9: Infinitesimal element in polar coordinate system

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty} \mathrm{e}^{-x^2} = \int_{-\infty}^{\infty} \mathrm{e}^{-y^2} \\
\therefore \quad I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{e}^{-y^2} dx dy \\
&= \int_0^{2\pi} d\theta \int_0^{\infty} \mathrm{e}^{-r^2} r dr \\
&= 2\pi \int_0^{\infty} r\mathrm{e}^{-r^2} = 2\pi \left[-\frac{1}{2}\mathrm{e}^{-r^2}\right]_0^{\infty} = \pi \\
\therefore \quad I &= \sqrt{\pi}
\end{aligned}
$$

Thus

$$\eta(x,t) = \frac{A\sqrt{\pi}}{d} e^{-\frac{(x-c_g t)^2}{4d^2}} e^{i(k_0 x - \omega_0 t)} \tag{29}$$

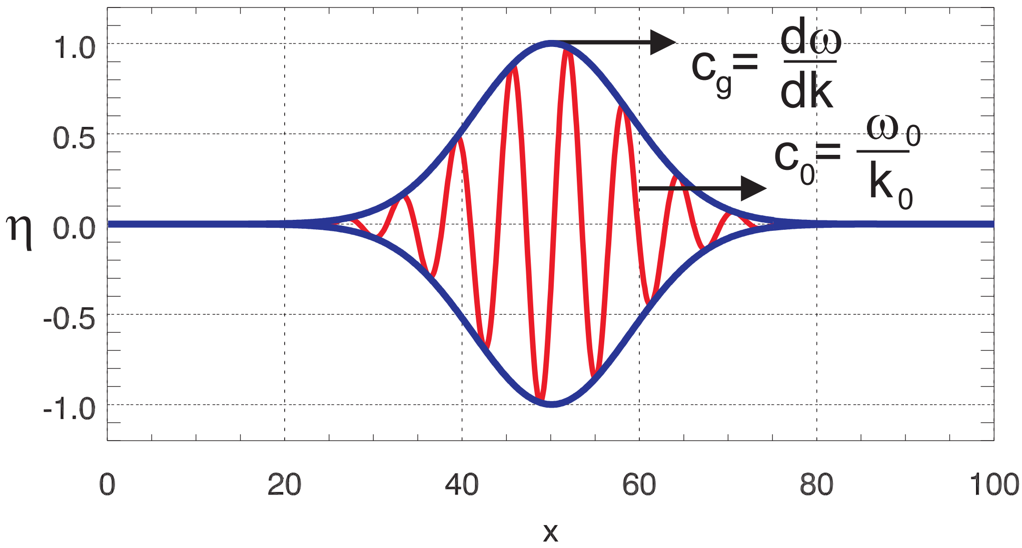

Figure 10: Wavepacket

4.6 The Doppler effect

• It is convenient here[5] to consider another general phenomenon connected with waves, namely the changes in frequency of waves sent out by a moving source and perceived by a stationary obsever. Consider sound waves for sound waves for definiteness.

We shall work in terms of the actual frequency ν and the wavelength λ, where

$$\nu = \frac{\omega}{2\pi}, \qquad \lambda = \frac{2\pi}{k}, \qquad c = \nu\lambda \tag{30}$$

As seen in the sketch, in a time t the source emits νt waves. For a stationary source these occupy a length $\nu t\lambda$, whereas, for a source moving with speed u towards the observer, the wavelength changes to λ' and the νt waves occupy a distance $\nu t\lambda'$. Thus

$$\nu t\lambda = \nu t\lambda' + ut \quad \Rightarrow \quad \lambda' = \lambda - \frac{u}{\nu}$$

$$\lambda' = \lambda(1 - \frac{u}{c}) \tag{31}$$

See bottom sketch on Handout.

[5]But not logical since it is more relevant to sound waves and radio waves than to surface waves than to surface waves on water!

As a result the observer measures the frequency of the waves as ν' where $\nu'\lambda' = c = \nu\lambda$. Thus

$$\nu' = \frac{\nu c}{c-u} \tag{32}$$

Example

An observer at rest notices that the frequency of the sound waves from a car appears to drop from 281 Hz to 257 Hz as the car passes. Given that the speed of sound is 330 ms^{-1}, estimate the speed of the car.

From Eq. (32) $\Rightarrow$

$$281 = \frac{\nu}{1-\frac{u}{c}}, \quad 257 = \frac{\nu}{1+\frac{u}{c}} \Rightarrow \quad \frac{281}{257} = \frac{1+\frac{u}{c}}{1-\frac{u}{c}}$$

$\Rightarrow$

$$\frac{u}{c} = \frac{24}{538} \quad \Rightarrow \quad u \approx 14.7\,\mathrm{m\ s^{-1}} \quad (\text{About} \quad 33\,\text{mph})$$

4.7 Particle paths in surface waves

Consider a particle whose equilibrium position is (x_0, z_0). Suppose its position at time t is $(x_0 + X(t), z_0 + Z(t))$, where the time means of X and Z will be chosen to be zero. Then

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= \left.\frac{\partial \phi}{\partial x}\right|_{(x_0+X, z_0+Z)} \approx \left.\frac{\partial \phi}{\partial x}\right|_{(x_0, z_0)} \\
&= \frac{kg\eta_0 \cosh k(z_0+h)}{\omega \cosh kh} \sin(kx_0 - \omega t) \\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= \left.\frac{\partial \phi}{\partial z}\right|_{(x_0+X, z_0+Z)} \approx \left.\frac{\partial \phi}{\partial z}\right|_{(x_0, z_0)} \\
&= -\frac{kg\eta_0 \sinh k(z_0+h)}{\omega \cosh kh} \cos(kx_0 - \omega t)
\end{aligned} \tag{33}$$

using Eq. (17). Thus, integrating and ensuring zero time means:

$$
\begin{aligned}
X &= \frac{kg\eta_0}{\omega^2}\frac{\cosh k(z_0+h)}{\cosh kh}\cos(kx_0-\omega t) \\
&= \eta_0\frac{\cosh k(z_0+h)}{\sinh kh}\cos(kx_0-\omega t) \\
Z &= \frac{kg\eta_0}{\omega^2}\frac{\sinh k(z_0+h)}{\cosh kh}\sin(kx_0-\omega t) \\
&= \eta_0\frac{\sinh k(z_0+h)}{\sinh kh}\sin(kx_0-\omega t)
\end{aligned} \tag{34}
$$

using the first of Eq. (16). It follows on eliminating $\cos(kx_0-\omega t)$ and $\sin(kx_0-\omega t)$ that

$$
\frac{X^2}{a^2}+\frac{Z^2}{b^2}=1 \quad \text{where} \begin{cases} a = \frac{\eta_0\cosh k(z_0+h)}{\sinh kh}, \\ b = \frac{\eta_0\sinh k(z_0+h)}{\sinh kh} \end{cases} \tag{35}
$$

Thus the particle paths are ellipses. As $z_0\to -h$, $b\to 0$, $a\to \eta_0/\sinh kh \Rightarrow$ rectilinear motion in direction of $0x$.

GROUP VELOCITY

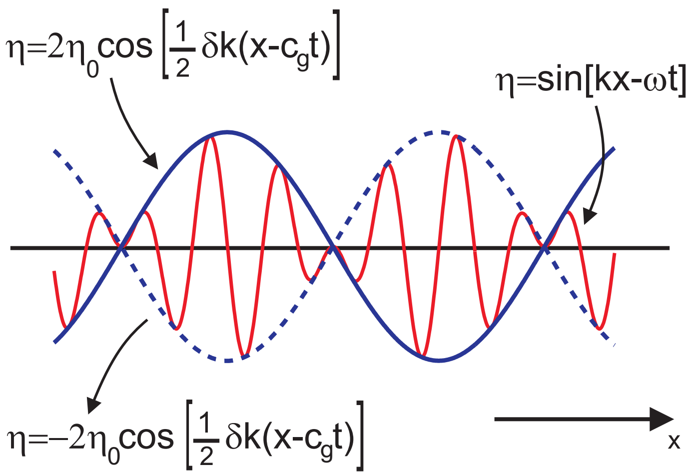

Figure 11: Sketch for Eq. (4.26).

DOPPLER EFFECT

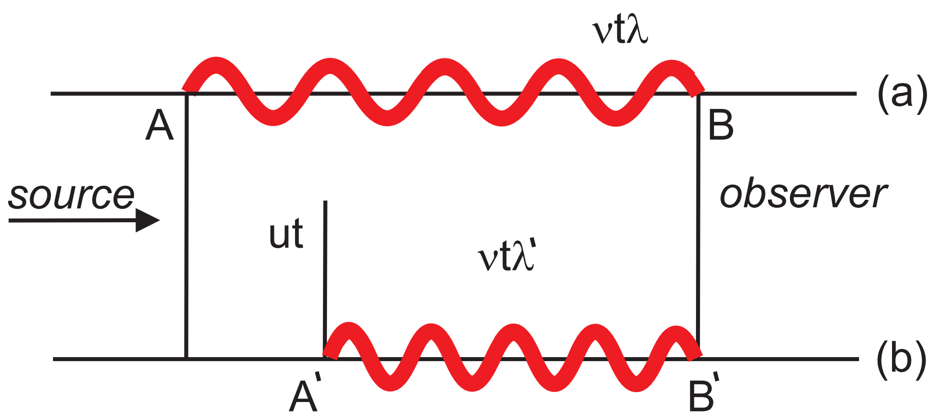

Figure 12: (a) Waves when source is stationary; (b) Waves when source is mooving. Sketch for Eq. (4.31).

5 Modelling Traffic Flow

5.1 Quasi-linear first-order PDEs

• We consider only two independent variables (x, y) and an unknown $z = z(x, y)$ satisfying a first order PDE. This PDE is quasi-linear if it is linear in its highest order terms, i.e.

$$z_x = \frac{\partial z}{\partial x} \quad \text{and} \quad z_y = \frac{\partial z}{\partial y}.$$

Thus

$$\begin{aligned} zz_x + z_y &= 0 \quad \text{is quasi-linear (and non-linear)} \\ (z_x)^2 + z_y &= 0 \quad \text{is not quasi-linear.} \end{aligned}$$

The most general first-order quasi-linear PDE is:

$$Pz_x + Qz_y = R \tag{1}$$

where

$$P = P(x, y, z), \; Q = Q(x, y, z), \; R = R(x, y, z) \tag{2}$$

are given continuous functions.

• Consider the family of curves in the (x, y) plane satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q}{P} \quad \text{or} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{P}{Q} \quad \text{or} \quad \frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}.$$

Suppose z is known at a point $A(x, y)$. There is one curve Γ_A of this family through A, and along Γ_A

$$\mathrm{d}z = z_x \mathrm{d}x + z_y \mathrm{d}y = \left(z_x + \frac{Q}{P} z_y \right) \mathrm{d}x = \frac{R}{P} \mathrm{d}x \qquad (3)$$

using Eq. (1). Hence $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{R}{P}$ along Γ_A, and so:

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q} = \frac{\mathrm{d}z}{R}. \qquad (4)$$

Eqs. (4) are known as the associated equations for Eq. (1), and are equivalent to Eq. (1). For let each term in Eq. (4) be $\mathrm{d}s$, so $\mathrm{d}x = P\mathrm{d}s$, $\mathrm{d}y = Q\mathrm{d}s$, $\mathrm{d}z = R\mathrm{d}s$. Substitute in Eq. (3) to get

$$R\mathrm{d}s = P z_x \mathrm{d}s + Q z_y \mathrm{d}s \Rightarrow P z_x + Q z_y = R,$$

i.e. Eq. (1).

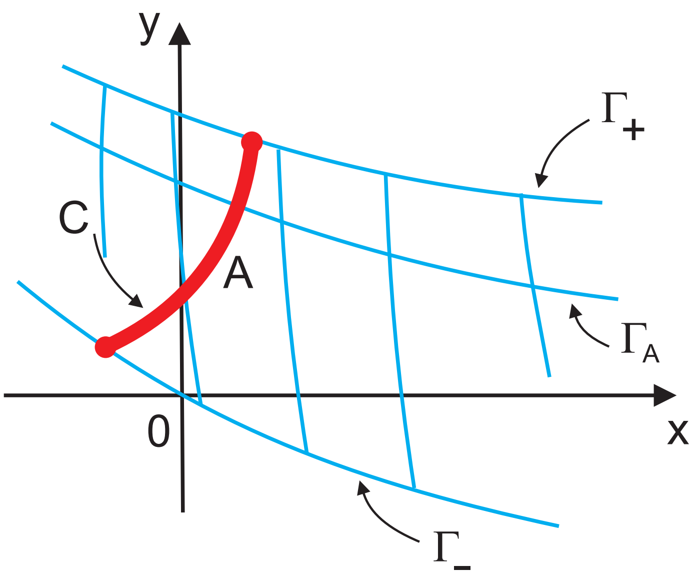

Figure 1: Characteristic curves

• Suppose z is given along a curve C in the (x, y) plane. Through each point A on C, we can continue the solution along Γ_A in both directions provided Γ_A is not parallel to C, i.e. provided that, on C, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is nowhere equal to $\frac{Q}{P}$. The curves Γ_A are known as the characteristics. Provided the characteristics do not intersect, we obtain a region bounded by Γ_+ and Γ_- within which z is known. If $\frac{Q}{P}$ is independent of z the characteristics are independent of the boundary conditions. In particular $\frac{Q}{P}$ is independent of z for a linear PDE. If P and Q are constants, then the characteristics are parallel straight lines.

Example 1

Solve $z_x - z_y = 1$ with $z = x^2$ on $y = 0$.

Solution

The associated equations are

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-1} = \frac{\mathrm{d}z}{1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -1, \ \frac{\mathrm{d}z}{\mathrm{d}x} = 1.$$

Thus the characteristics are $x + y = \alpha$ and on the characteristics $z - x = \beta$.

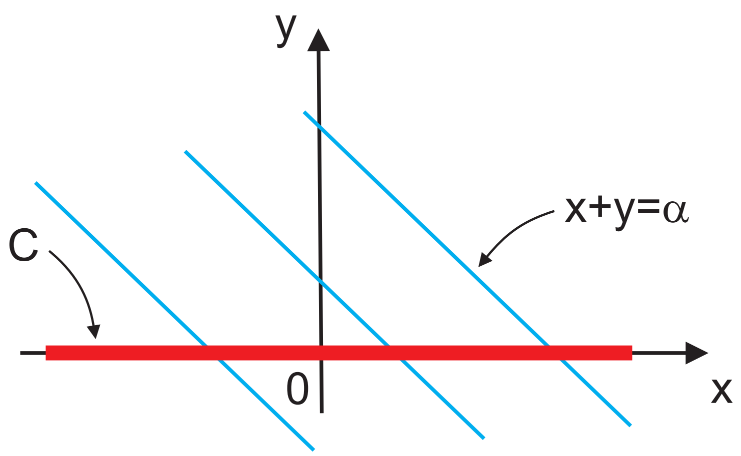

Figure 2: Characteristics of Example 1

The curve C is $y = 0$ and each point on C is intercepted by exactly one characteristic. We can proceed in two ways.

(A) When $y = 0$, $x + y = \alpha \Rightarrow x = \alpha$ and $z = \alpha^2$.

Hence from $z = x + \beta \Rightarrow \beta = \alpha^2 - \alpha$.

Thus $z = x + \left(\alpha^2 - \alpha\right)$ on $x + y = \alpha$.

Eliminate α to get $z = x + \left((x + y)^2 - (x + y)\right) \Rightarrow$

$$z = (x + y)^2 - y.$$

(B) Since $z - x$ is constant when $x + y$ is constant $\Rightarrow$

$z - x = f(x + y)$ for some function f. But $z = x^2$ when $y = 0 \Rightarrow x^2 - x = f(x)$.

Thus $z = x + \left((x + y)^2 - (x + y)\right) \Rightarrow$

$$z = (x + y)^2 - y$$

Example 2

Solve $yz_x + xz_y = z$ with $z = x^3$ on $y = 0$ and $z = y^3$ on $x = 0$.

Solution

The associated equations are

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}y}{x} = \frac{\mathrm{d}z}{z} \Rightarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \Rightarrow x^2 - y^2 = \alpha$$

are characteristics, where α =const. Then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{z}{\sqrt{(x^2-\alpha)}} \Rightarrow \frac{\mathrm{d}z}{z} = \frac{\mathrm{d}x}{\sqrt{(x^2-\alpha)}}$$

$\Rightarrow$

$$\begin{aligned} \ln z &= \ln\left(x + \sqrt{\left(x^2-\alpha\right)}\right) + \beta' \\ z &= \beta\left(x + \sqrt{\left(x^2-\alpha\right)}\right) \end{aligned}$$

on a characteristic, where $\beta = \mathrm{e}^{\beta'}$ =const.

$$\begin{aligned} \alpha = x^2 - y^2 \Rightarrow z &= \beta\left(x + \sqrt{\left(x^2 - x^2 + y^2\right)}\right) \\ &= \underline{\beta(x+y)} \quad \text{or} \quad \underline{\beta(x-y)}. \end{aligned}$$

Case 1: $z = \beta(x+y)$

In this case we find, that as in Ex 1 (B) above,

$$\frac{z}{x+y} \quad \text{is constant when} \quad x^2 - y^2 \quad \text{is constant.}$$

The GS is therefore $z = (x+y)f\left(x^2 - y^2\right)$, and it remains to determine f.

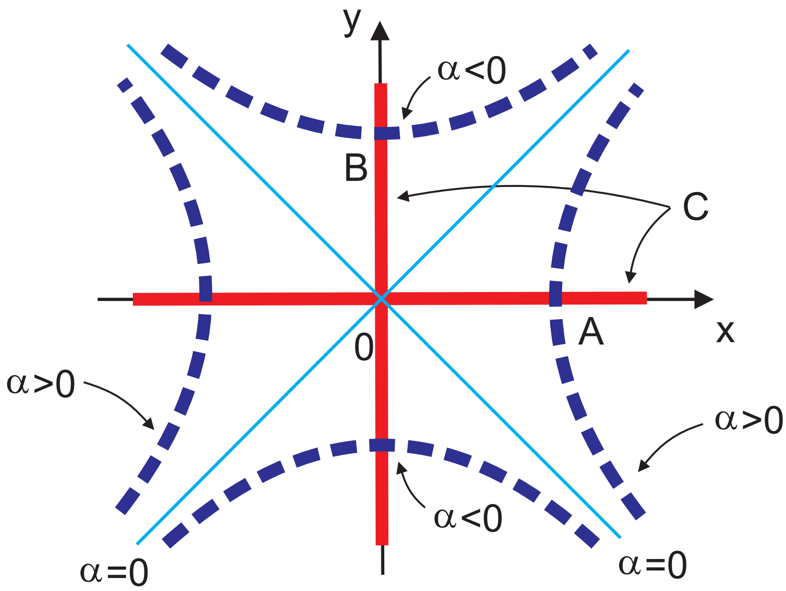

Figure 3: Characteristics of Example 2

We are given z on both axes. At the common point O, $z = 0$ from both prescriptions. The characteristics through O are $x = \pm y$ $(\alpha = 0)$ and on these $z = 0$. The result is obviously symmetric about both axes.

Suppose $\alpha > 0$.

Consider $A(\alpha_1^{\frac{1}{2}}, 0)$ at which

$$z = x^3 = \alpha_1^{\frac{3}{2}}.$$

So from the GS $\Rightarrow$

$$\alpha_1^{\frac{3}{2}} = \alpha_1^{\frac{1}{2}} f(\alpha_1) \Rightarrow f(\alpha_1) = \alpha_1 \Rightarrow (x+y)(y^2 - x^2)$$

for $x^2 > y^2$.

Suppose $\alpha < 0$.

Consider $B(0, (-\alpha_2)^{\frac{1}{2}})$ at which

$$z = y^3 = (-\alpha_2)^{\frac{3}{2}}.$$

So from the GS $\Rightarrow$

$$\begin{aligned} (-\alpha_2)^{\frac{3}{2}} &= (-\alpha_2)^{\frac{1}{2}}) f(\alpha_2) \Rightarrow f(\alpha_2) = -\alpha_2 \\ \Rightarrow z &= (x+y)(y^2 - x^2) \quad \text{for} \quad x^2 < y^2 \end{aligned}$$

In summary

$$z = \begin{cases} (x+y)(x^2 - y^2) & \text{for} \quad x^2 > y^2 \\ 0 & \text{for} \quad x^2 = y^2 \\ (x+y)\left(y^2 - x^2\right) & \text{for} \quad x^2 < y^2 \end{cases} \tag{5}$$

Case 2: $z = \beta(x - y)$

In this case it can be similarly deduced that the GS of the PDE is

$$z = (x - y)g(x^2 - y^2).$$

However, this GS gives

$$\begin{aligned} z_x &= g\left(x^2 - y^2\right) + 2x(x - y)g'\left(x^2 - y^2\right) \\ z_y &= -g\left(x^2 - y^2\right) - 2y(x - y)g'\left(x^2 - y^2\right) \end{aligned}$$

Therefore

$$yz_x + xz_y = -(x - y)g(x^2 - y^2) = -z$$

which is not our original PDE, therefore we dismiss this second case, $z = \beta(x - y)$, as spurious solution.

5.2 Some properties of characteristics

• We begin by considering Eq. (5). It is clear that z is everywhere continuous, and that z_x, z_y are everywhere continuous except possibly on the lines

$$x = \pm y \qquad (x^2 - y^2) = 0.$$

From Eq. (5) we find

$$\begin{aligned} x^2 > y^2 : \; z_x &= (x^2 - y^2) + 2x(x+y) = (x+y)(3x - y) \\ z_y &= (x^2 - y^2) - 2y(x+y) = (x+y)(x - 3y) \end{aligned}$$

$$\begin{aligned} x^2 < y^2 : \; z_x &= (y^2 - x^2) - 2x(x+y) = (x+y)(y - 3x) \\ z_y &= (y^2 - x^2) + 2y(x+y) = (x+y)(3y - x). \end{aligned}$$

Thus as $x \longrightarrow -y$ from either side, $z_x \longrightarrow 0$ and $z_y \longrightarrow 0$. Hence z_x and z_y are continuous on $x + y = 0$.

Hovever, as $x \longrightarrow y$, z_x jumps from $+4x^2$ $(x > y)$ to $-4x^2$ $(x < y)$, and z_y jumps from $-4x^2$ $(x > y)$ to $+4x^2$ $(x < y)$. Thus z_x and z_y are discontinuous across the characteristic $x = y$.

We investigate the possibility of discontinuities in z_x and z_y for Eq.(1), but we shall suppose z is everywhere continuous. (The standard terminology is that we are looking for weak discontinuities whereas discontinuities in z itself are strong discontinuities).

Suppose C is approached from + and −. Then

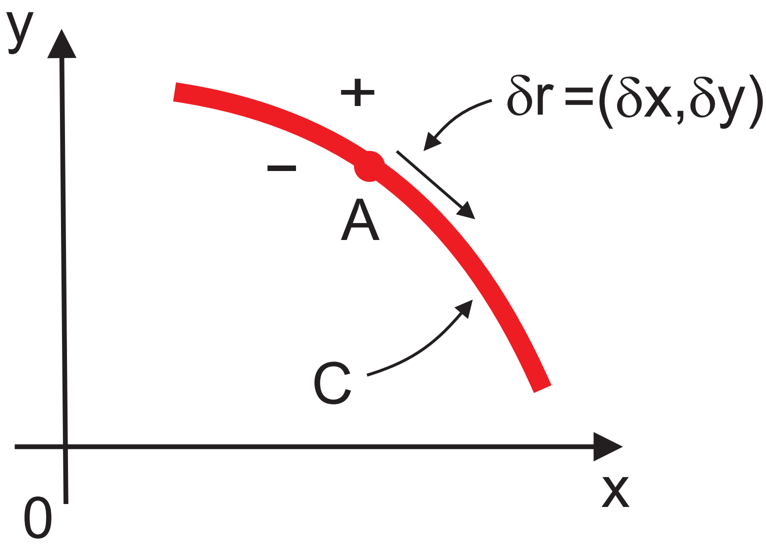

Figure 4: Jump across characteristics

$$\delta z^{+} = \frac{\partial z^{+}}{\partial x}\delta x + \frac{\partial z^{+}}{\partial y}\delta y$$
$$\delta z^{-} = \frac{\partial z^{-}}{\partial x}\delta x + \frac{\partial z^{-}}{\partial y}\delta y$$

where $(\delta x, \delta y)$ is along C. Subtract.

Because z is continuous, $\delta z^+ = \delta z^-$. Hence

$$\delta x \left[\frac{\partial z}{\partial x}\right]_-^+ + \delta y \left[\frac{\partial z}{\partial y}\right]_-^+ = 0. \qquad (6)$$

where the square brackets denote the jump in the expression across C.

Since Eq. (1) is satisfied on both sides and since, by hypothesis, P, Q, R are continuous

$$P \left[\frac{\partial z}{\partial x}\right]_-^+ + Q \left[\frac{\partial z}{\partial y}\right]_-^+ = 0. \qquad (7)$$

The necessary condition for

$$\left[\frac{\partial z}{\partial x}\right]_-^+ \neq 0 \quad \text{and} \quad \left[\frac{\partial z}{\partial y}\right]_-^+ \neq 0 \quad \text{is} \quad \frac{\delta x}{P} = \frac{\delta y}{Q},$$

i.e.

$$\frac{\mathrm{d}x}{P} = \frac{\mathrm{d}y}{Q}. \qquad (8)$$

Thus C must be a characteristic Γ_A.

• When Q/P is independent of z, the characteristics are independent of the boundary conditions.

When Q/P depends on z, different boundary conditions produce different sets of characteristics.

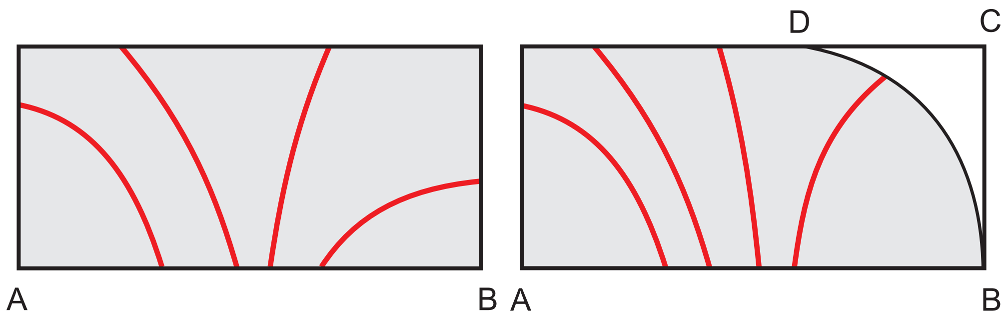

Figure 5: (i) z determined throughout rectangle; (ii) z not determined in BCD

We can also consider situations in which z itself is discontinuous at a point on the boundary. Then the shape of the characteristics (in the case when Q/P depends on z) will change discontinuously at that point. Qualitatively, there are two possibilities:

- In (i) there appear to be two characteristics through a point, whereas
- in (ii) there is a region containing no characteristics.

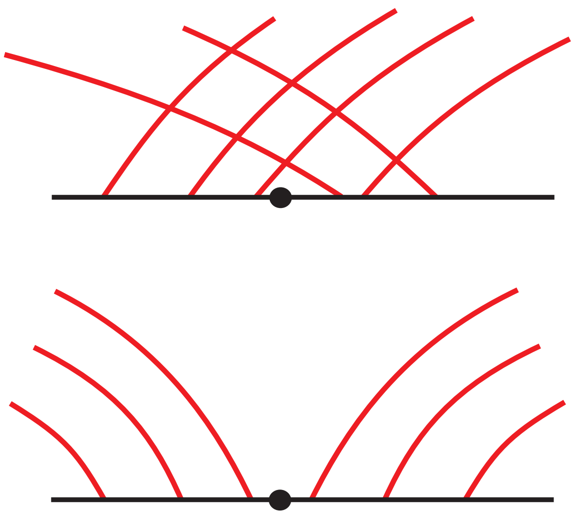

Figure 6: Characteristics leading to (i) shocks; or to (ii) centred fans (also called rarefication shocks)

Again, qualitatively these two situations will be relevant to our models of traffic flow [(i) leads to shocks, (ii) leads to centred fans - see later].

• We can obtain similar situations to (i) and (ii), but even without initial discontinuities in the slopes of the characteristics. The following example will connect well with our models of traffic flow.

Example

Solve

$$\rho_t + \rho\rho_x = 0$$

with $\rho = f(x)$ on $t = 0$. Consider two special cases:

$$\textit{Case 1}: \quad f(x) = 0 \quad (x < 0), \quad f(x) = x \quad (0 \le x < 1), \quad f(x) = 1 \quad (x \ge 1);$$

$$\textit{Case 2}: \quad f(x) = 0 \quad (x < 0), \quad f(x) = -x \quad (0 \le x < 1), \quad f(x) = -1 \quad (x \ge 1).$$

The associated equations Eq. (4) are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\rho} = \frac{\mathrm{d}\rho}{0}$$

where the last is to be interpreted as $\mathrm{d}\rho = 0$.

Thus

$$\rho = \alpha, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \alpha \Rightarrow x - \alpha t = \beta \quad \text{are characteristics.}$$

Now consider the characteristic through $x = \xi$ on $t = 0$. On this characteristic $\rho = \alpha = f(\xi)$. Thus

$$x = f(\xi)t + \xi, \;\; \rho = f(\xi) \tag{9}$$

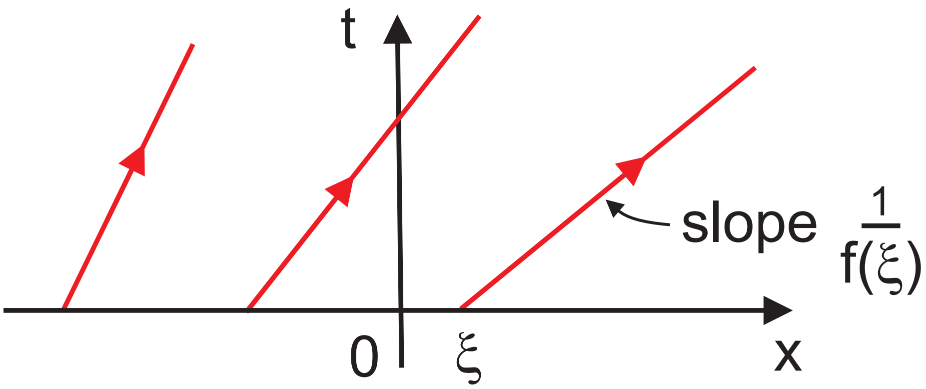

Figure 7: Characteristics for Example

Alternatively, from Eq. (9)

$$x = \rho t + \xi$$

so that we can write Eq. (9) implicitly as

$$\rho = f(x - \rho t) \tag{10}$$

Case 1: From Eq. (10)

$$\rho = 0 \quad (x < 0), \qquad \rho = x - \rho t \quad (0 \leq x - \rho t < 1)$$

$\Rightarrow$

$$\rho = \frac{x}{1+t} \quad (0 \leq x < 1+t), \qquad \rho = 1 \quad (x \geq 1+t),$$

i.e.

$$\rho = \begin{cases} 0 & (x < 0) \\ x/(1+t) & (0 \leq x < 1+t) \\ 1 & (x \geq 1+t) \end{cases} \tag{11a}$$

Case 2: Likewise,

$$\rho = \begin{cases} 0 & (x < 0) \\ x/(1-t) & (0 \leq x < 1-t) \\ -1 & (x \geq 1-t) \end{cases} \tag{11b}$$

The solution Eq. (11b) breaks down at $t = 1$; as the sketch on the hand-out shows, t he characteristics intersect at $t = 1$ in Case 2 and the profile of ρ against x becomes triple-valued (but this cannot occur in reality).

5.3 Model of traffic flow

We assume:

1. One lane of traffic in direction of Ox with no overtaking.
2. We can define a local car density $\rho = \rho(x, t)$ as the number of cars per unit length of road.
3. The local car velocity $v(x, t)$ is a function of ρ alone, i.e.

$$v = v(\rho) \tag{12}$$

The meaning of Eq. (12) is that each driver adjusts his, her or its speed to local conditions exclusively, whereas most drivers look ahead and adjust speed where appropriate. These assumptions give a car flowrate $q(\rho)$ with

$$q(\rho) = \rho v(\rho) \tag{13}$$

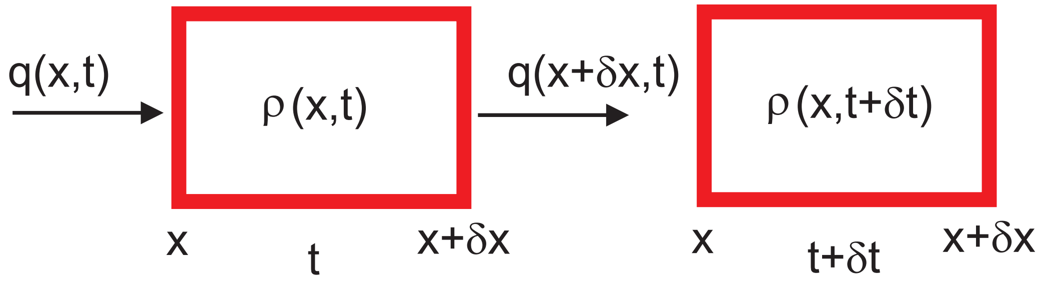

Figure 8: Car flow

Consider two values of x, viz. x_1, x_2 with $x_1 \leq x \leq x_2$.

At time t, the number of cars in this interval is

$$\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x.$$

The rate of change of this must be the net flowrate, viz.

$$\frac{\partial}{\partial t}\left\{\int_{x_1}^{x_2} \rho(x,t)\mathrm{d}x\right\} = [q(x,t)]_{x_2}^{x_1} \tag{14}$$

If $x_1 = x$, $x_2 = x + \delta x$, Eq. (14) becomes

$$\frac{\partial}{\partial t}\rho\delta x = -\frac{\partial q}{\partial x}\delta x$$

$\Rightarrow$

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{15}$$

- We need to model $v(\rho)$.

We assume there is a maximum possible density P with essentially "bumper-to-bumper" traffic. When $\rho = P$, we assume $v(\rho)$ in Eq. (12) is zero.

We also assume $v(\rho)$ decreases as ρ increases, with a maximum of V when $\rho = 0$.

These assumptions are shown schematically...

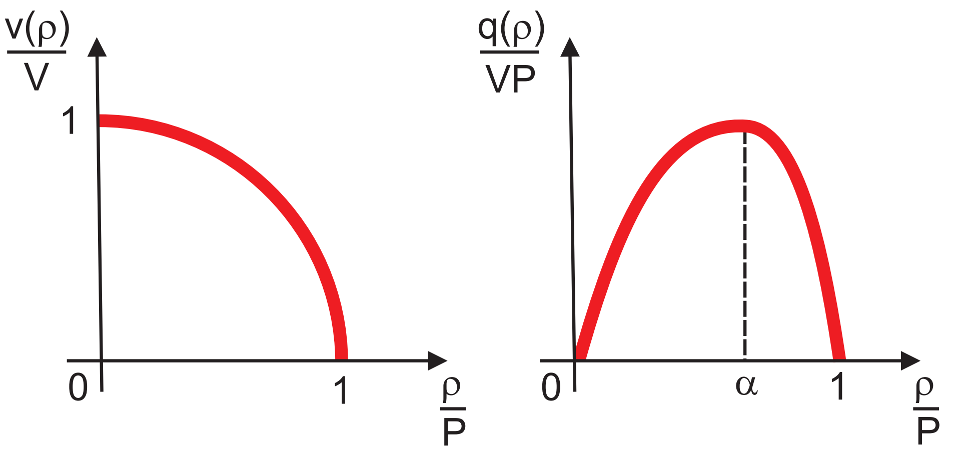

Figure 9: Modelling car flow

With Eq. (13), Eq. (15) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho v(\rho)\right)\frac{\partial \rho}{\partial x} = 0$$

or

$$\begin{aligned} \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} &= 0, \\ c(\rho) &= \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho v(\rho)\right) = v(\rho) + \rho v'(\rho) \end{aligned} \tag{16}$$

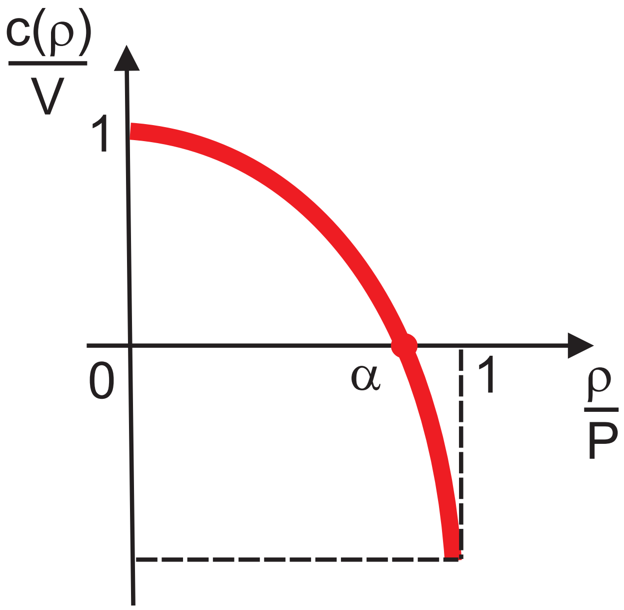

Figure 10: Model of car flow

The assumptions made about $v(\rho)$ give a $c(\rho)$ which is monotonic decreasing and negative for $\rho/P > \alpha$, where α is the value of ρ/P for which $q(\rho)$ is a maximum.

5.4 Small amplitude disturbances from a uniform state

• Before studying the full non-linear problem, it is intructive to consider a simpler one. Suppose that there is almost a uniform state with $\rho = \rho_0$ and

$$\rho = \rho_0 + \rho' \text{ with } |\rho'| \ll \rho_0. \tag{17}$$

Linearise Eq. (16) - as with sound waves earlier - to get

$$\frac{\partial \rho'}{\partial t} + c\left(\rho_0\right) \frac{\partial \rho'}{\partial x} = 0. \tag{18}$$

Either

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c\left(\rho_0\right)} = \frac{\mathrm{d}\rho'}{0} \Rightarrow$$

$$\rho' = \text{const.} \quad \text{on} \quad x - c(\rho_0)t = \text{const.}$$

Or put $\xi = x - c\left(\rho_0\right) t \Rightarrow$

$$\frac{\partial \rho'}{\partial x} = \frac{\partial \rho'}{\partial \xi}, \quad \left(\frac{\partial \rho'}{\partial t}\right)_x = \left(\frac{\partial \rho'}{\partial t}\right)_\xi - c\left(\rho_0\right)\left(\frac{\partial \rho'}{\partial \xi}\right) \Rightarrow$$

$$\left(\frac{\partial \rho'}{\partial t}\right)_\xi = 0.$$

Thus the GS of Eq. (18) is

$$\rho' = f\left\{x - c\left(\rho_0\right) t\right\}. \tag{19}$$

- The characteristics of Eq. (18) are the straight lines

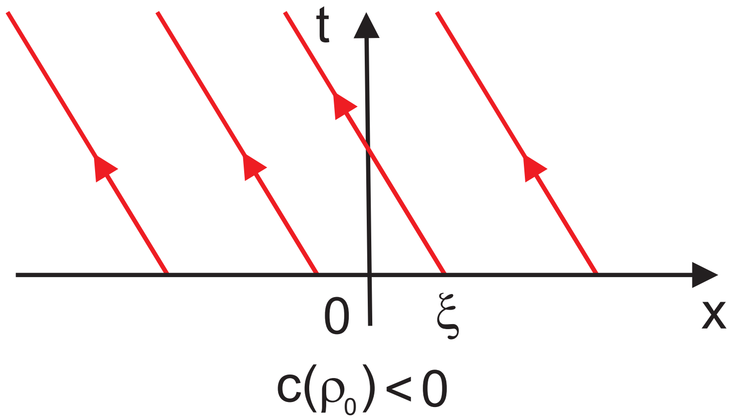

Figure 11: Characteristics of car flow are straight lines

$$x = \xi + c\,(\rho_0)\,t. \tag{20}$$

Eq. (19) shows that ρ' is constant on each characteristic.

Eq. (19) represents a wave travelling to the right with speed $c\,(\rho_0)$. If $\rho_0/P > \alpha \Rightarrow c(\rho_0) < 0$.

This is a kinematic wave; $c\,(\rho_0)$ is the speed of the disturbance, not of the cars.

This explains a common phenomenon on a busy road when a sudden increase in density reaches you from ahead with no apparent reason.

5.5 The initial value problem for Eq. (16)

• We wish to solve Eq. (16) subject to the initial condition

$$\rho(x,0) = f(x) \tag{21}$$

By the earlier methods - see especially the Example in § (5.2) - ρ is constant on the characteristics

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\rho} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \rho.$$

Since ρ is constant on a characteristic, the characteristics are straight.

$\Rightarrow$

Thus, if $c\{f(\xi)\} = F(\xi)$, the solution can be written for $t \geq 0$

$$\rho = f(\xi) \quad \text{on the straight line} \quad x = \xi + F(\xi)t. \tag{22}$$

Example

Suppose

$$v(\rho) = \frac{V}{P}(P - \rho) \tag{23}$$

and that $\rho(x, 0) = f(x)$ satisfies

$$\rho = \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(\rho_L - \rho_R)\tanh\frac{x}{L} \tag{24}$$

where ρ_L, ρ_R and L are constants. Discuss the solution given by Eq. (22) when

$$(i) \quad \rho_L > \rho_R, \quad \text{and} \quad (ii) \quad \rho_L < \rho_R.$$

Solution

From flow model Eq. (23) it follows

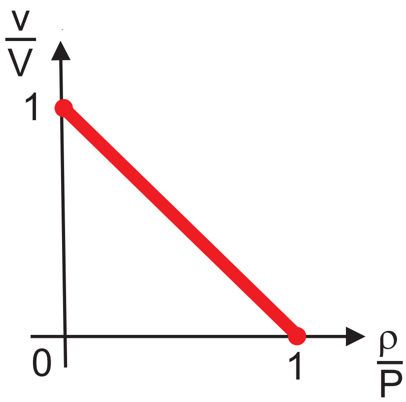

Figure 12: (a) Car flow $v(\rho) = V(P - \rho)/P$

$$q(\rho) = \frac{V}{P}\left(P\rho - \rho^2\right), \quad c(\rho) = \frac{V}{P}(P - 2\rho). \qquad (25)$$

Note also that

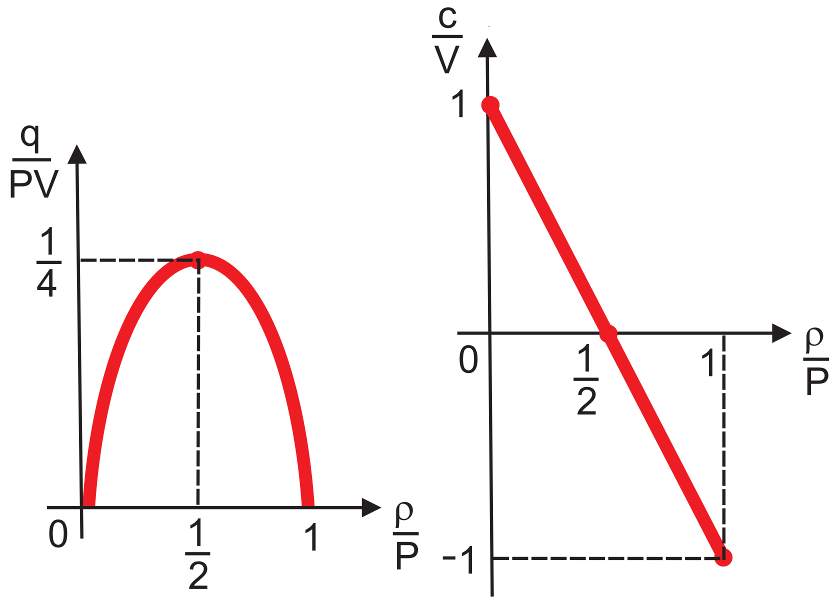

Figure 12: (b) Car flow flux, and (c) speed of disturbance

$$\rho \to \rho_L \quad \text{as} \quad \frac{x}{L} \to -\infty$$

and

$$\rho \to \rho_R \quad \text{as} \quad \frac{x}{L} \to +\infty.$$

Also

$$\begin{aligned} F(\xi) &= c\{f(\xi)\} \qquad (26) \\ &= \frac{V}{P}\left[P - \rho_L - \rho_R + (\rho_L - \rho_R)\tanh\left(\frac{\xi}{L}\right)\right] \end{aligned}$$

Case (i): $\rho_L > \rho_R \Rightarrow F'(\xi) > 0 \quad \text{for} \quad \forall\xi$

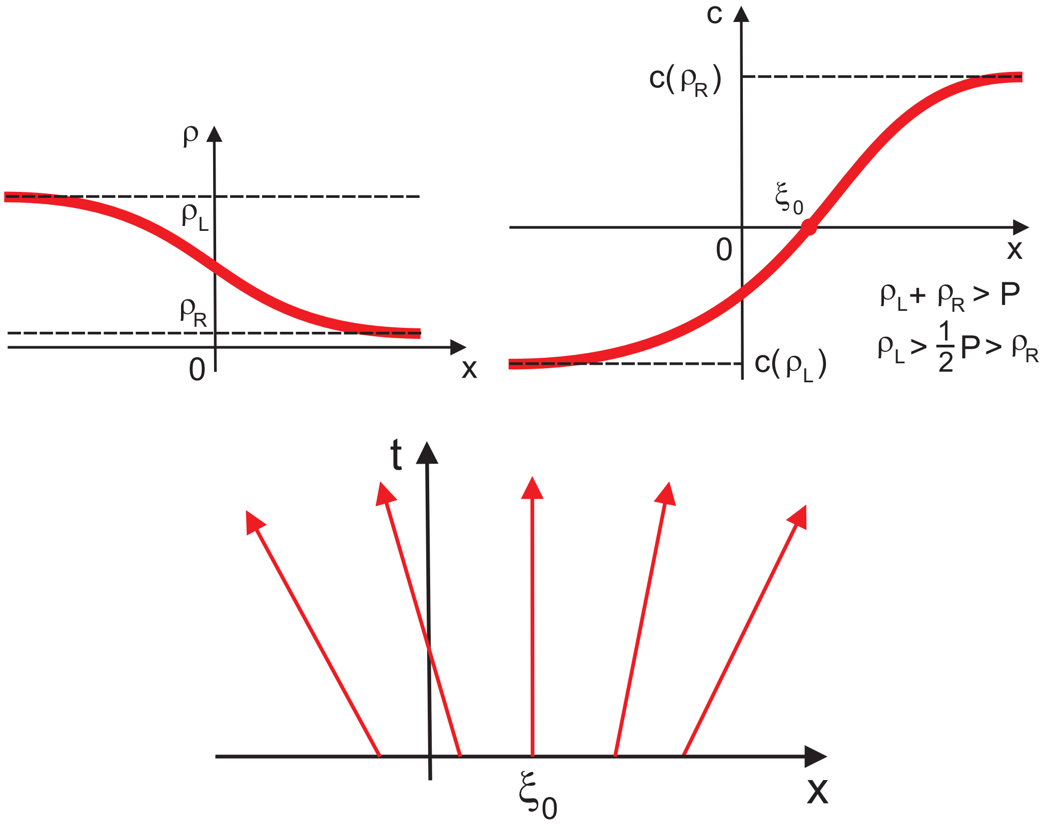

Figure 13: (a) Car flow, (b) profile of speed of disturbance, and (c) characteristics for Case (i). Note, that ρ is constant on each characteristic

Case (*ii*): $\rho_L < \rho_R \Rightarrow F'(\xi) < 0 \quad \text{for} \quad \forall \xi$

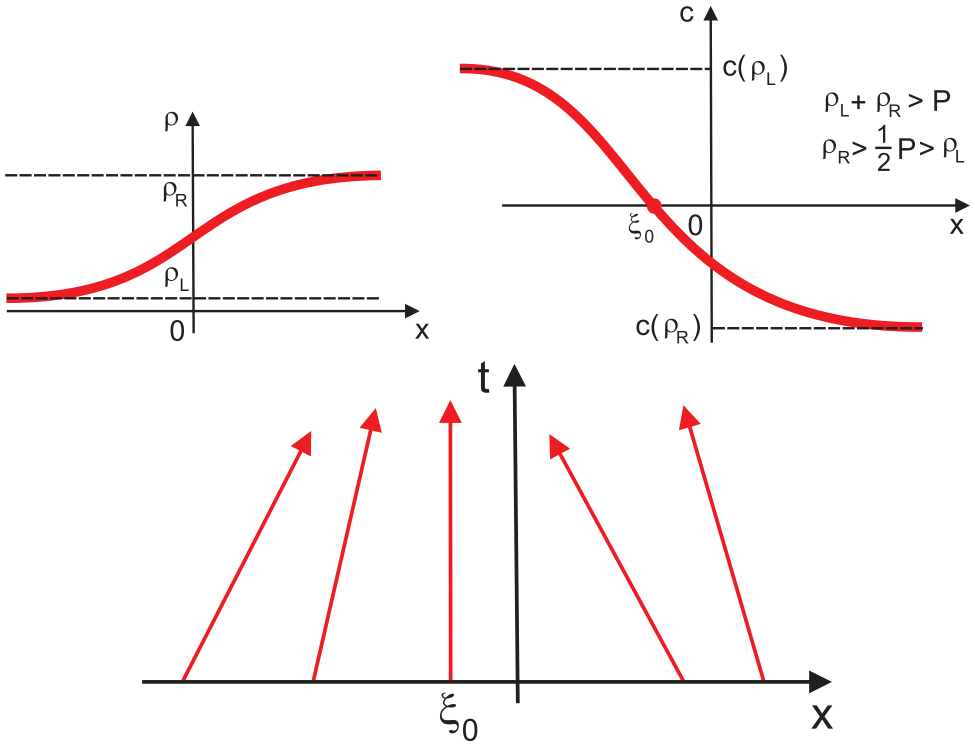

Figure 14: (a) Car flow, (b) profile of speed of disturbance, and (c) characteristics for Case (*ii*). Note, from (c) that characteristics eventually intersect

Characteristics eventually intersect $\Rightarrow$ problem becomes ill-posed.

If two characteristics intersect, any enclosed characteristic must meet one of them at an earlier time $\Rightarrow$ earliest intersection must be between neighbouring characteristics.

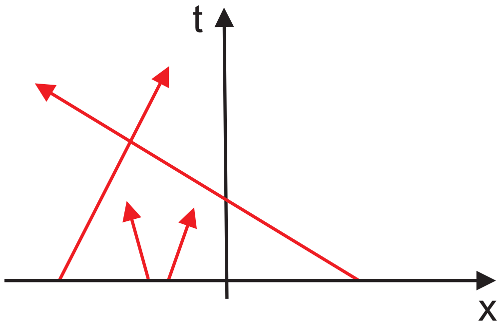

Figure 15: Intersecting characteristics

Suppose these are

$$\left.\begin{aligned} x &= \{\xi + F(\xi)t\} \\ \\ x &= \{(\xi + \partial\xi) + F(\xi + \partial\xi)t\} \\ &= \{\xi + F(\xi)t\} + \{1 + F'(\xi)t\}\,\partial\xi \end{aligned}\right\} \Rightarrow$$

$$\therefore 1 + F'(\xi)t = 0. \tag{27}$$

We get solutions of Eq. (27) with $t > 0$ only if $\exists\,\xi$ with $F'(\xi) < 0$. [Thus for $\rho_L > \rho_R$ there are no intersections and the solution given by Eq. (22) applies for $\forall\; t \geq 0$.]

The first positive t satisfying Eq. (27) occurs when

$$t = T_{\min} = \frac{1}{\underset{-\infty<\xi<\infty}{\text{Max}} \{-F'(\xi)\}}. \tag{28}$$

While Eqs. (27) and (28) are general, we can calculate $T_{\min}$ in our particular case when Eq. (26) holds.

We find

$$-F'(\xi) = \frac{V}{P}(\rho_R - \rho_L)\frac{1}{L}\text{sech}^2\left(\frac{\xi}{L}\right)$$

$\Rightarrow$

$$\max\{-F'(\xi)\} = \frac{V(\rho_R - \rho_L)}{PL}$$

when $\xi = 0$. Then Eq. (28) gives

$$T_{\min} = \frac{PL}{V(\rho_R - \rho_L)}. \tag{29}$$

5.6 Shocks

• We can understand in another way why there is trouble when $\rho_L < \rho_R$. With $c'(\rho) < 0$, low densities propagate forward relative to high densities. The profile of ρ against x inevitably steepens as t increases and has a vertical section at $t = T_{\min}$. Were we to continue, the profile would develop the triple-valued shape - clearly unacceptable since ρ must be a single valued quantity.

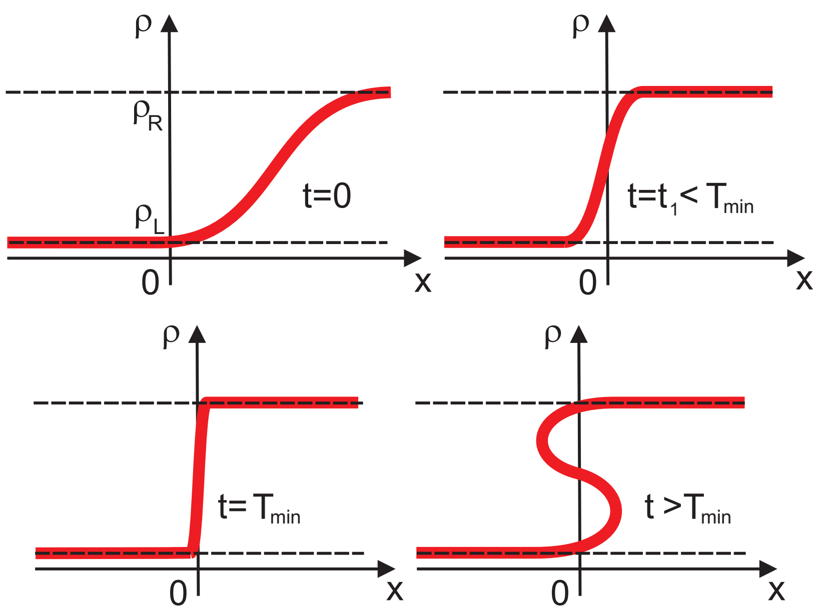

Figure 16: Development of shock

• Instead the wave breaks, and the model must be extended. A consistent extension conserves cars but allows discontinuities in ρ to occur across a shock.

We cannot have characteristics crossing one another. Instead the picture is as shown schematically.

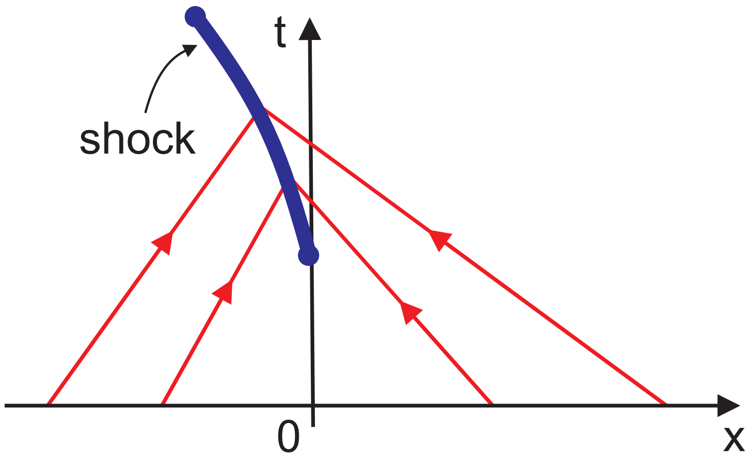

Figure 17: Shock front and characteristics

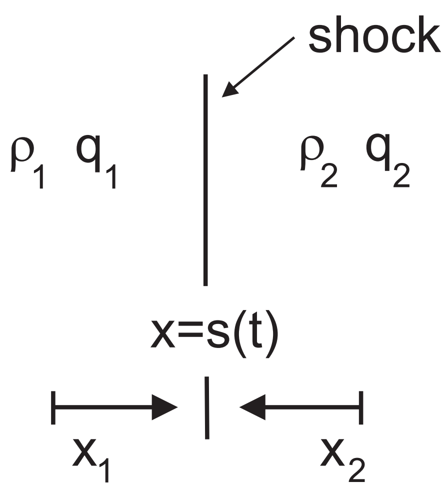

Figure 18: Quantities at a shock front

From Eq. (14) $\Rightarrow$

$$\frac{\partial}{\partial t}\int_{x_1}^{s(t)} \rho \mathrm{d}x + \frac{\partial}{\partial t}\int_{s(t)}^{x_2} \rho \mathrm{d}x = q_1 - q_2$$

$$\begin{aligned}
\text{LHS} &= \underbrace{\int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t}\mathrm{d}x}_{\to 0 \text{ as } x_1 \to s_-} + \frac{1}{\partial t}\left\{\int_{x_1}^{s(t+\delta t)} - \int_{x_1}^{s(t)}\right\}\rho \mathrm{d}x \\
&\quad + \underbrace{\int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t}\mathrm{d}x}_{\to 0 \text{ as } x_2 \to s_+} + \frac{1}{\partial t}\left\{\int_{s(t+\delta t)}^{x_2} - \int_{s(t)}^{x_2}\right\}\rho \mathrm{d}x \\
&= \frac{1}{\partial t}\left\{\int_{s(t)}^{s(t)+\delta t} \rho_1 \mathrm{d}x - \int_{s(t)}^{s(t)+\delta t} \rho_2 \mathrm{d}x\right\} \\
&= \dot{s}\left(\rho_1 - \rho_2\right)
\end{aligned}$$

by the mean value theorem.

$\Rightarrow$

$$\dot{s}\left(\rho_1 - \rho_2\right) = \left(q_1 - q_2\right) \tag{30a}$$

$$\dot{s} = \frac{q_1 - q_2}{\rho_1 - \rho_2}. \tag{30b}$$

• The position of the shock is fixed by the need to conserve cars. Without a shock the curve of ρ against x would become (unacceptably) triple-valued. We insert the shock so that the shaded areas are equal, thus ensuring that $\int \rho \mathrm{d}x$ = number of cars is unchanged. This is known as (Whitham's) equal area rule.

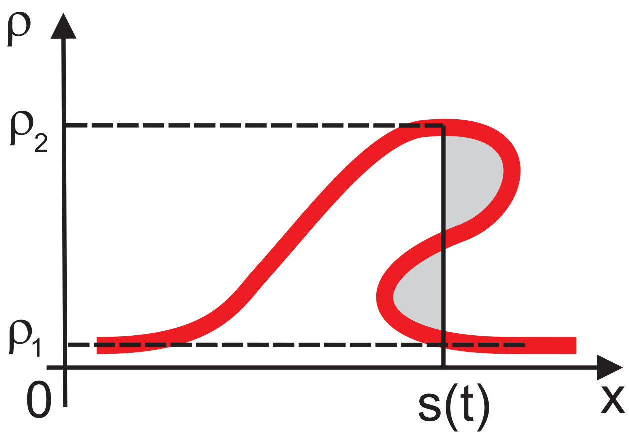

Figure 19: Shock fitting: Whitham's equal area rule

• As an application consider what happens as cars approach a stationary queue behind a red traffic light so that $\rho_R = P$, $\rho_L < P$. On meeting the queue cars stop and the lengthening of the queue is achieved by a shock wave propagating backwards.

5.7 The Riemann problem

• We wish to consider the case when we solve Eqs. (16) and (21) where there is a discontinuity in $f(x)$. It will be sufficient to consider the simplest possible case, viz.

$$\rho(x,0) = f(x) = \begin{cases} \rho_L & (x < 0) \\ \\ \rho_R & (x > 0) \end{cases} \tag{31}$$

• Then Eq. (22) gives the solution as

$$\begin{aligned} \rho = \rho_L \quad &\text{on } x = \xi + c\,(\rho_L)\,t \quad (\xi < 0) \\ \rho = \rho_R \quad &\text{on } x = \xi + c\,(\rho_R)\,t \quad (\xi > 0) \end{aligned} \tag{32}$$

Consider first the case $\rho_L > \rho_R$. The characteristic diagram is easy to draw...

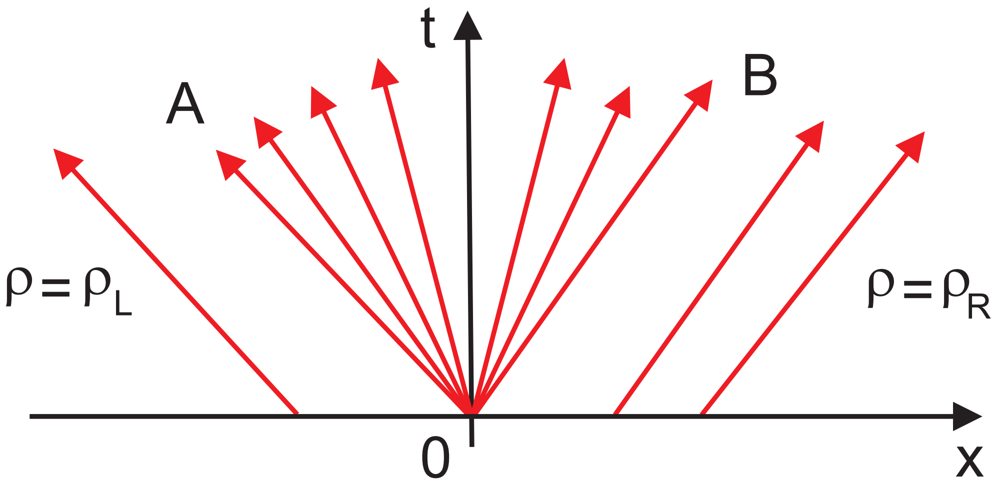

Figure 20: Fan of characteristics

They are either parallel to OA with slope $c\,(\rho_L)$; to the left of OA, $\rho = \rho_L$. Or they are parallel to OB with slope $c(\rho_R)$; to the right of OB, $\rho = \rho_L$. But what happens in OAB?

• The problem arises because of the discontinuity and can be solved by considering a limit process in which ρ takes all the values from ρ_R to ρ_L, and all the characteristics go through the origin. Thus

$$\rho = k \qquad (\rho_R < \rho < \rho_L)$$

on $x = c(k)t$.

The solution is therefore:

$$\rho = \begin{cases} \rho_L & : \quad x < c\,(\rho_L)\,t \\ k \text{ on } x = c(k)t & : \quad c\,(\rho_L) < c(k) < c\,(\rho_R) \\ \rho_R & : \quad x > c\,(\rho_R)\,t \end{cases} \qquad (33)$$

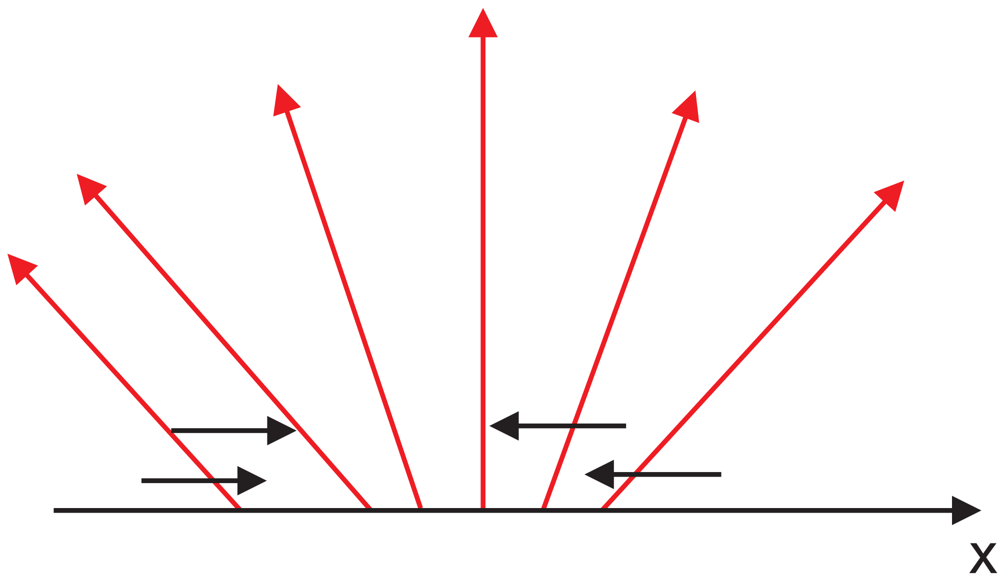

Figure 21: Centered fan or expansion fan corresponding or rarefication wave

The characteristic diagram is augmented by a centred fan or an expansion fan or expansion wave or rarefication wave.

• Conversely, when $\rho_L < \rho_R$ the characteristic diagram shows immediately trouble whose only resolution is a shock starting from $t = 0$ with speed, given by Eq. (30b) as U, where

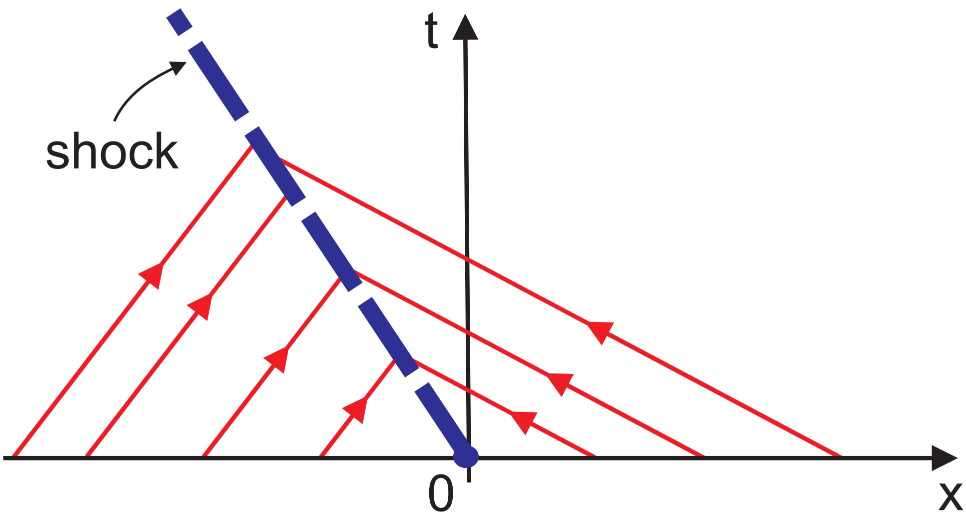

Figure 22: Schematic shock with speed U

$$U = \frac{q\left(\rho_L\right) - q\left(\rho_R\right)}{\left(\rho_L - \rho_R\right)} \tag{34}$$

5.8 Additional refinements

• The model assumptions leading to Eq. (16) are too simple. One extension is to suppose that q is a function of the density gradient $\partial\rho/\partial x$ as well as ρ, thus allowing drivers to reduce their speed to account for an increasing density ahead. A simple assumption is to take

$$q = Q(\rho) - \nu\rho_x \tag{35}$$

where ν is a positive constant. Thus q decreases if ρ_x is positive, i.e. it there is an increasing density ahead. Use of Eq. (35) in Eq. (15) gives

$$\rho_L + c(\rho)\rho_x = \nu\rho_{xx}, \;\; c(\rho) = q'(\rho) \tag{36}$$

• Seek solutions of Eq. (36) of the form

$$\rho = \rho(X) \qquad X = x - Ut \tag{37}$$

where U is a constant still to be determined. Substitution in Eq. (36) $\Rightarrow$

$$-U\rho'(X) + c(\rho)\rho'(X) = \nu\rho''(X).$$

Since $c(\rho) = Q'(\rho)$ we have

$$Q(\rho) - U\rho + C = \nu\rho'(X) \tag{38}$$

where C is a constant. Suppose $\rho \longrightarrow \rho_L$ as $X \longrightarrow -\infty$ and $\rho \longrightarrow \rho_R$ as $X \longrightarrow +\infty \Rightarrow$

$$Q(\rho_L) - U\rho_L + C = Q(\rho_R) - U\rho_R + C = 0,$$

$\Rightarrow$

$$U = \frac{Q(\rho_R) - Q(\rho_L)}{\rho_R - \rho_L} \tag{39}$$

This is exactly Eq. (30b) but (for the moment) in a different context.

• Since $\rho \longrightarrow \rho_L$ as $X \longrightarrow -\infty$ and $\rho \longrightarrow \rho_R$ as $X \longrightarrow +\infty$, $\rho'(x) = 0$ at $\rho = \rho_L$ and $\rho = \rho_R$. We suppose ρ_L and ρ_R are simple zero's of

$$Q(\rho) - U\rho + C,$$

and more precisely we shall suppose $\rho_L < \rho_R$ and

$$Q(\rho) - U\rho + C = \alpha(\rho - \rho_L)(\rho_R - \rho) \qquad (\alpha > 0) \tag{40}$$

With $\alpha > 0$,

$$c(\rho) = Q'(\rho) = \alpha\,(\rho_R - \rho) - \alpha\,(\rho - \rho_L)$$

and

$$c'(\rho) = \alpha\,(\rho_L - \rho_R) < 0.$$

We can always approximate $Q(\rho)$ by a quadratic. Then Eq. (38) becomes

$$\nu \frac{\mathrm{d}\rho}{\mathrm{d}X} = \alpha\,(\rho - \rho_L)\,(\rho_R - \rho)$$

with solution

$$\left(\frac{\rho_R - \rho}{\rho - \rho_L}\right) = \left(\frac{\rho_R - \rho_0}{\rho_0 - \rho_L}\right) \mathrm{e}^{-\frac{X}{L}}, \qquad L = \frac{\nu}{\alpha\,(\rho_R - \rho_L)} \tag{41}$$

where $\rho = \rho_0$ at $X = 0$. We note that $\rho \to \rho_L$ as $X \to -\infty$ and that $\rho \to \rho_R$ as $X \to +\infty$ as required.

The transition between $\rho \sim \rho_L$ and $\rho \sim \rho_R$ occupies a thickness of order L.

As L diminishes, i.e. as ν diminishes for fixed α and $(\rho_R - \rho_L)$ the transition takes place more sharply $\Rightarrow$ a shock is approached.

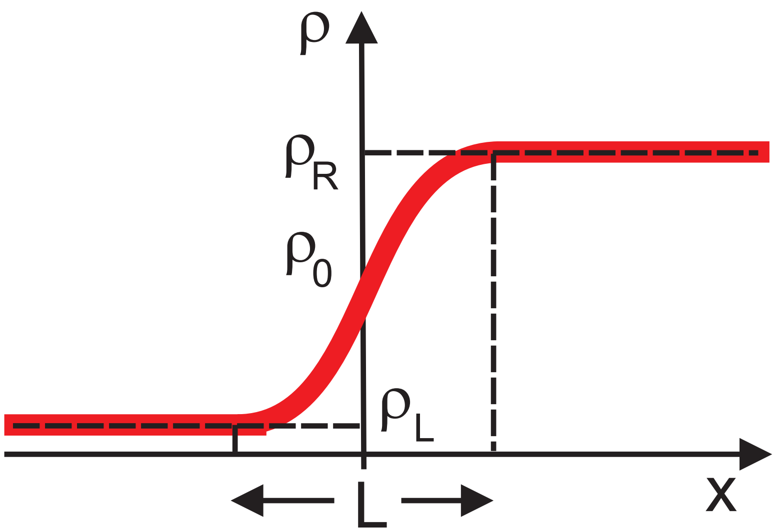

Figure 23: Development of shock front

• The model in this section can be taken further in the case when Eq. (40) holds.

Multiply Eq. (36) by $c'(\rho) \Rightarrow$

$$c'(\rho)\rho_t + c(\rho)c'(\rho)\rho_x - \nu c'(\rho)\rho_{xx}$$

$$\therefore \qquad c_t + cc_x = \nu\frac{\partial^2 c}{\partial x^2} - \nu c''(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2$$

because

$$\frac{\partial c}{\partial x} = c'(\rho)\frac{\partial \rho}{\partial x}$$

and therefore

$$\frac{\partial^2 c}{\partial x^2} = c''(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2 + c'(\rho)\frac{\partial^2 \rho}{\partial x^2}.$$

In the case when $Q(\rho)$ is quadratic, i.e. Eq. (40) holds, $c''(\rho) = 0$ since $c(\rho) = Q'(\rho)$. Thus

$$c_t + cc_x = \nu c_{xx}. \tag{42}$$

This is known as Burger's equation and, remarkably, it can be solved explicitly by means of the transformation

$$c = -2\nu\frac{\phi_x}{\phi} \tag{43}$$

discovered independently by E. Hopf (1950) and J.D.Cole (1951). Use of Eq. (43) transforms Eq. (42) into the standard linear equation (after one integration w.r.t. x):

$$\phi_t = \nu\phi_{xx}. \tag{44}$$

It can be shown that this is also consistent with the shock structure.

• A second refinement is that there is a time lag in driver response. One way of handling this is to take Eq. (35) and deduce from it that $v = q/\rho$ satisfies

$$v = V(\rho) - \frac{\nu}{\rho}\rho_x, \qquad V(\rho) = \frac{Q(\rho)}{\rho}. \tag{45}$$

Then regard this as a velocity which the driver tries to achieve. The acceleration of the car is

$$\frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

[see Notes after § (4.3)] and the model is

$$v_t + vv_x = -\frac{1}{\tau}\left\{v - V(\rho) + \frac{\nu}{\rho}\rho_x\right\}, \tag{46}$$

where τ is a measure of the response time. Eq. (46) is to be solved together with Eq. (15), i.e.

$$\rho_t + (\rho v)_x = 0. \tag{47}$$

THE END OF SECTION A