SECTION C: SOLITONS

Course text: *Solitons: an introduction*
P.G. Drazin & R.S. Johnson
Cambridge University Press, 1989

You will find the lecture notes at
http://www.shef.ac.uk/~robertus/kul/

Outline of the course

1. Introduction
2. Waves of permanent form
3. Scattering & inverse scattering
4. The inverse scattering transform for the KdV equation
5. Conservation Laws
6. The Lax method

1. Introduction

Linear PDEs

Consider the PDE

$$Lu = 0$$

where $L = L(\partial/\partial x, \partial/\partial t)$ and $u = u(x, t)$.

If L is linear, *i.e.* if

$$L(au + bw) = aLu + bLw \tag{1}$$

$\forall$ constants a and b, and $\forall$ 'well behaved' functions u and w, then

$$Lu = 0 \ \& \ Lw = 0 \Longrightarrow L(au + bw) = 0 \tag{3}$$

i.e. if u and w are solutions of

$$Lv = 0 \tag{5}$$

then so is any linear combination of u and w.

This is called ***the superposition principle***

Example: the wave equation

Take

$$L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \tag{6}$$

We have that if

$$u(x,t) = f(x-ct) + g(x+ct) \tag{8}$$

for $f, g \in C^2(-\infty, \infty)$, then

$$Lu = 0 \quad \forall x, t \tag{9}$$

If $u(x,0) = F(x)$ and $u_t(x,0) = G(x)$ then

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)d\xi \qquad (11)$$

is the unique solution to $Lu = 0$ with the given initial conditions.

This is called ***D'Alambert solution.***

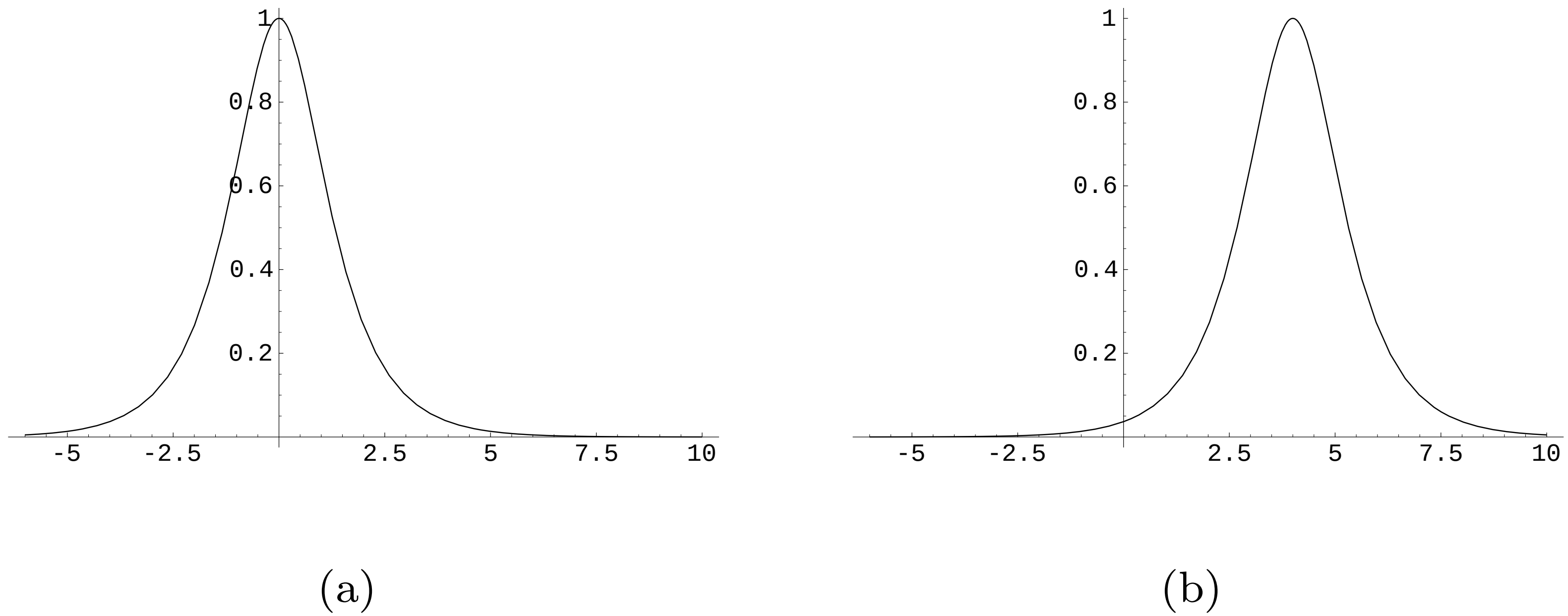


Figure 1: The D'Alambert solution $f(x - ct)$ of the wave equation at two different times

Consider $f(x-ct)$ and $g(x+ct)$ separately. They obey the equation

$$u_t + cu_x \quad \text{and} \quad u_t - cu_x.$$

respectively.

This follows from the identity

$$L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \equiv \left(\frac{\partial}{\partial t} \pm c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} \mp c \frac{\partial}{\partial x} \right) \quad (13)$$

Normal modes

Suppose that $D(i\partial/\partial t, -i\partial/\partial x)$ is a linear operator and

$$Du = 0. \tag{14}$$

Then try (*i.e.* guess that there exists) a solution of the form

$$u(x,t) = ae^{i(kx-\omega t)}$$

for constant ***wavenumber*** k and ***frequency*** ω.

$u(x,t)$ represents a sinusoidal wave of ***length*** $2\pi/k$, ***period*** $2\pi/\omega$ and ***phase velocity*** $c = \omega/k$.

*A solution of a linear equation depending exponentially on time is called **normal mode***

Because $D(i\partial/\partial t, -i\partial/\partial x)$ is linear, if $u(x,t)$ is a normal mode

$$D(i\partial/\partial t, -i\partial/\partial x)u = D(i(i\omega), -i(ik))u = 0, \tag{15}$$

which implies

$$D(\omega, k) = 0.$$

*This is called **dispersion relation**.*

Example: the wave equation

Consider again the wave equation

$$Du \equiv \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0. \tag{16}$$

Try $u \propto e^{i(kx-\omega t)}$

$$(-i\omega)^2 u - c^2 (ik)^2 u = 0 \tag{17}$$

that is

$$D(\omega, k) = c^2 k^2 - \omega^2 = 0 \tag{18}$$

$$\omega = \pm ck. \tag{19}$$

*$\pm c$ is called **phase velocity** of mode.*

If the phase velocity is not constant different modes travel at different velocities and eventually the wave (packet) ***disperses***.

The simplest dispersive equation is

$$u_t + u_x + u_{xxx} = 0.$$

Let us try $u \propto e^{i(kx-\omega t)}$, then

$$D(\omega, k) = -i\omega + ik + (ik)^3 = 0 \tag{20}$$

$$\omega = k - k^3 \tag{21}$$

$$c = 1 - k^2 \quad \leq 1. \tag{22}$$

Note that long waves (small k) travel faster.

If we add even derivatives to $u_t + u_x$, $\omega = f(k)$ is a complex function of k, and the wave ***dissipates***.

Consider

$$u_t + u_x - u_{xx} = 0,$$

then by trying $u \propto e^{i(kx-\omega t)}$

$$D(\omega, k) = -i\omega + ik - (ik)^2 = 0 \tag{23}$$

$$\omega = k - ik^2 \tag{24}$$

$$u(x,t) = e^{-k^2 t + ik(x-t)} \tag{25}$$

The above wave decays exponentially.

An initial value problem

Suppose that

$$D(i\partial/\partial t, -i\partial/\partial x)u = 0 \tag{26}$$

is a linear PDE of first order in $\partial/\partial t$ and $u(x, 0)$ is given.

Let us take the Fourier transform of $u(x, 0)$:

$$a(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, 0)e^{-ikx}dx \tag{27}$$

and

$$u(x, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx}dk \tag{28}$$

Suppose that

$$D(\omega, k) = 0 \Longrightarrow \omega = f(k). \qquad (29)$$

Then it may be verified that

$$u(x,t) = \int_{-\infty}^{\infty} a(k) e^{i[kx - f(k)t]} dk. \qquad (31)$$

Wave packet and group velocity

A localized solution which is the superposition of waves of approximately the same length is called a ***wave packet****.*

The components have in general slightly different phase velocity $c = \omega/k$*, and therefore spread, i.e.* ***disperse****.*

It can be shown that asymptotically the wave packet moves with the ***group velocity****, which is defined by*

$$c_g = \frac{d\omega}{dk}. \tag{33}$$

It can be shown that any localized disturbance after a long time propagates at the group velocities, rather than phase velocities of its components. So physical properties, like energy, have velocity c_g ***not*** c*. It turns out that in general* $c_g \leq c$*.*

Example

Consider the following expression:

$$
\begin{aligned}
& a\cos(k_1 x - \omega_1 t) + a\cos(k_2 x - \omega_2 t) \\
&= 2a\cos\left[\frac{1}{2}\left(k_2 - k_1\right)x - \frac{1}{2}\left(\omega_2 - \omega_1 t\right)\right] \\
&\quad \times \cos\left[\frac{1}{2}\left(k_2 + k_1\right)x - \frac{1}{2}\left(\omega_2 + \omega_1\right)t\right] \\
&\sim 2a\cos\left[\frac{1}{2}\left(k_2 - k_1\right)\left(x - c_g t\right)\right]\cos\left[k_1\left(x - ct\right)\right] \\
&= A(\epsilon x, \epsilon t)\cos(k_1 x - \omega_1 t) \quad \text{as } k_2 \to k_1.
\end{aligned} \tag{34}
$$

Here $A(\epsilon x, \epsilon t) = 2a\cos\left[\epsilon\left(x - c_g t\right)\right]$, $\epsilon = (k_2 - k_1)/2$ is small and

$$
c_g = \lim_{k_2 \to k_1} \frac{\omega_2 - \omega_1}{k_2 - k_1} \quad \text{and} \quad c = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}. \tag{35}
$$

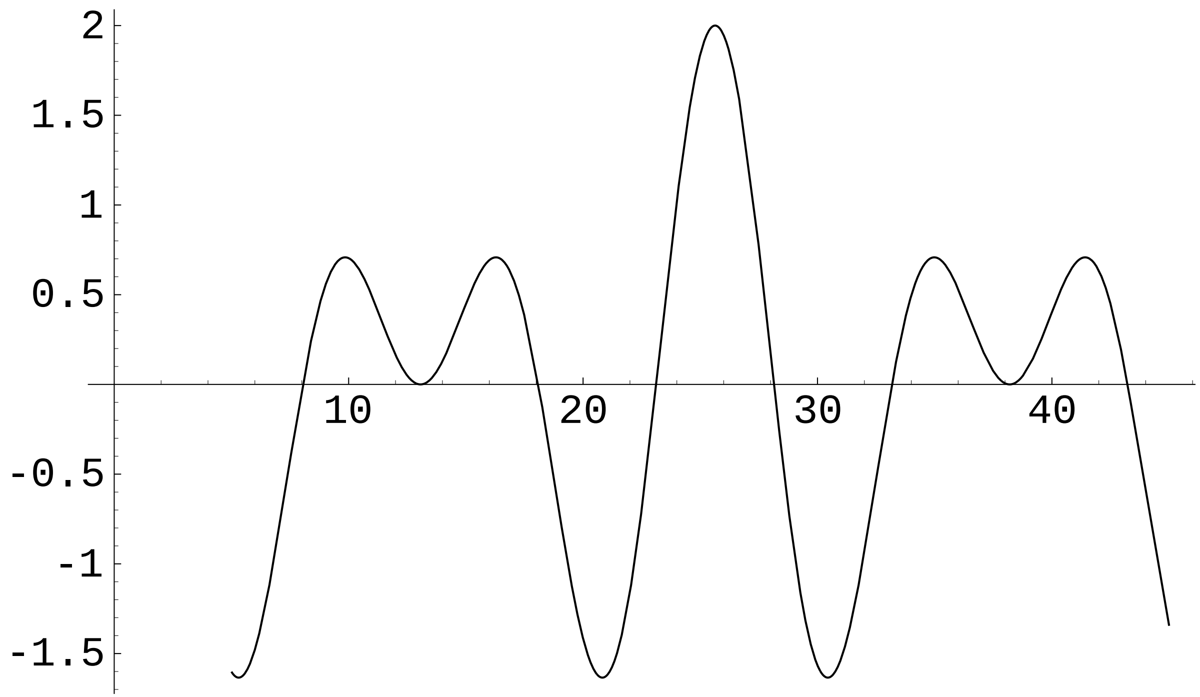


Figure 2: The ‘wave packet’ (34).

The method of characteristics and nonlinear waves

Consider again the equation $\boxed{u_x + cu_t = 0}$. Then

$$u(x,t) = \text{constant} \quad \forall t \tag{36}$$

on curves with equation $dx/dt = c$ in the (x,t)-plane, because

$$\frac{du}{dt} = u_t + \frac{dx}{dt}u_x = u_t + cu_x = 0. \tag{37}$$

Using this property, solution to $u_x + cu_t = 0$ can be easily constructed.

If

$$u(x, 0) = f(x) \tag{38}$$

for differentiable f, then

$$u(x, t) = f(x - ct).$$

Now consider the nonlinear equation

$$u_x + c(u)u_t = 0.$$

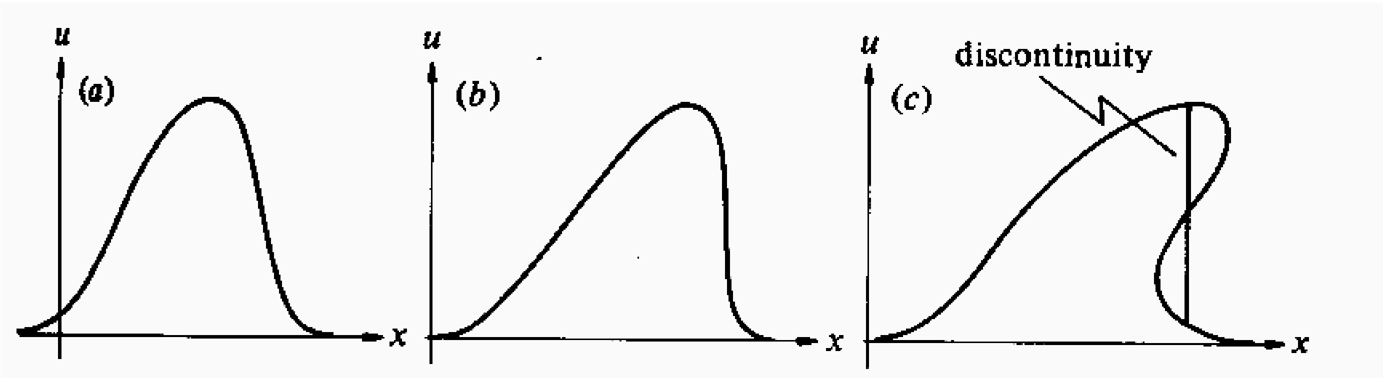
Generalizing the previous ideas, $u(x, t)$ is constant on curves given by the equation

$$\frac{dx}{dt} = c(u), \quad x(t) = c(u)t + \text{constant}. \tag{39}$$

If furthermore $u(x, 0) = f(x)$, the solution is given implicitly by

$$u(x, t) = f\,(x - c(u)t).$$

If $c(u)$ increases with u, then the greater u is the faster the velocity dx/dt so that an initial solution will steep, and may go on to ***break***.



The solution may be continued beyond breaking by invoking some additional hypothesis.

Example

A simple example of nonlinear equation which can have multivalued solution is

$$u_x + (1 + u)u_t = 0.$$

Using the method of characteristics, we have that the general solution is

$$u(x, t) = f\left(x - (1 + u)\, t\right),$$

where f is an arbitrary function.

Example

Now, consider

$$u_x + u u_t = 0.$$

If $u(x,0) = \cos \pi x$, then

$$u(x,t) = \cos\left[\pi\left(x - ut\right)\right] \tag{40}$$

gives $u(x,t)$ implicitly.

Where do shocks occur?

By differentiating $u(x,t) = \cos[\pi(x - ut)]$ w.r.t. x we have

$$u_x = -\pi(1 - u_x t)\sin[\pi(x - ut)] \tag{41}$$

$$u_x\{1 - \pi t\sin[\pi(x - ut)]\} = -\pi\sin[\pi(x - ut)]. \tag{42}$$

Shocks occur when $u_x = \infty$, which implies

$$t\pi\sin[\pi(x - ut)] = 1.$$

The first shock occurs when

$$t = 1/\pi \text{ and } x - ut = 1/2 + 2n \text{ for } n = 0, \pm 1, \pm 2, \ldots$$

i.e. where $u = 0, x = 1/2 + 2n$.

We may have ***nonlinearity and dissipation***, *e.g.*

$$u_t + (1+u)u_x - u_{xx} = 0$$

or ***nonlinearity and dispersion***,

$$u_t + (1+u)u_x + u_{xxx} = 0$$

Let us consider the transformations

$$1 + u \rightarrow u, \quad \alpha u, \quad t \rightarrow \beta t, \quad x \rightarrow \gamma x \tag{43}$$

for the equation in the blue box

we obtain

$$u_t + \frac{\alpha\beta}{\gamma} u u_x + \frac{\beta}{\gamma^3} u_{xxx} = 0. \qquad (44)$$

Choosing (for example) $\alpha = -6$, $\beta = \gamma = 1$ yields

$$u_t - 6 u u_x + u_{xxx} = 0.$$

This is known as ***Korteweg-de Vries (KdV) equation****.*

The discovery of Solitary waves.

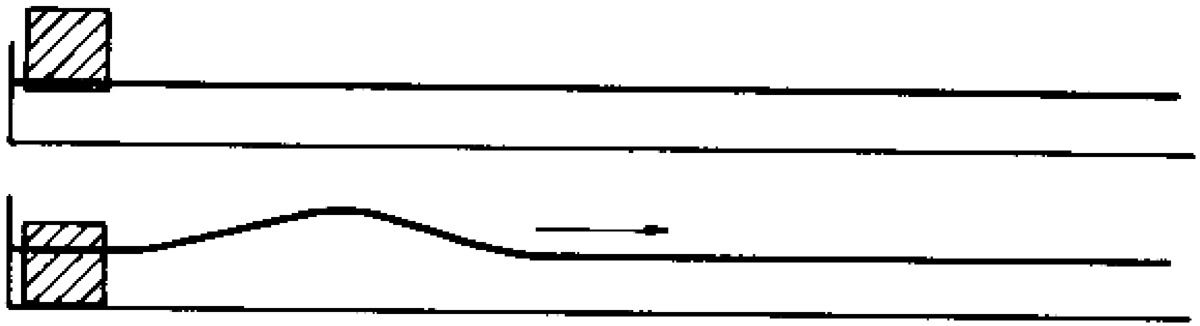


Figure 3: The diagram of Russel's experiment to generate solitary waves

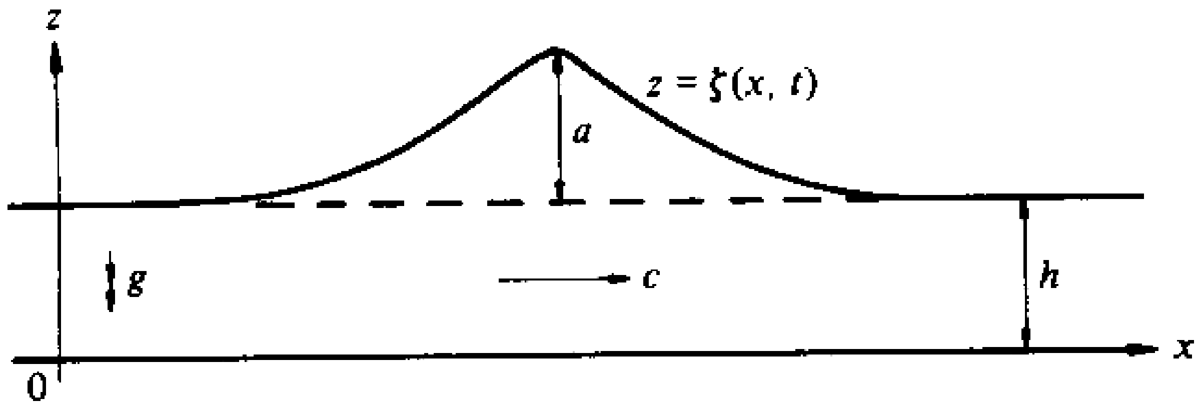


Figure 4: Parameters used in the description of solitary waves

- J. Scott Russel (1834):

$$c^2 = g(h+a) \tag{45}$$

- Boussinesq (1871), Rayleigh (1876):

$$\zeta(x,t) = a\,\mathrm{sech}^2\left[\beta\left(x - ct - x_0\right)\right]. \tag{46}$$

- Korteweg & de Vries (1895):

  Derived the equation for long weekly nonlinear water waves

$$\frac{\partial \zeta}{\partial t} = \frac{3}{2}\left(\frac{g}{h}\right)^{1/2}\left(\frac{2}{3}\epsilon\frac{\partial \zeta}{\partial \chi} + \zeta\frac{\partial \zeta}{\partial \chi} + \frac{1}{3}\sigma\frac{\partial^3 \zeta}{\partial \chi^3}\right) \tag{47}$$

The function $\zeta(\chi) = a\,\text{sech}^2\,(\beta\chi)$ is a solution to the previous equation provided

$$a = 4\sigma\beta^2 \quad \text{and} \quad \epsilon = -2\sigma\beta^2. \tag{48}$$

The coordinate χ is defined as

$$\chi = x - (gh)^{1/2}\left(1 - \frac{\epsilon}{h}\right)t. \tag{49}$$

Finally, the solitary wave solution becomes

$$\zeta(x,t) = a\,\text{sech}^2\left[\frac{1}{2}\left(\frac{a}{\sigma}\right)^{1/2}\left\{x - (gh)^{1/2}\left(1 + \frac{a}{2h}\right)t\right\}\right]. \tag{51}$$

$$c \sim (gh)^{1/2}\left(1 + \frac{a}{2h}\right) \tag{52}$$

Def.: *A **solitary wave** solution of a PDE*

$$N\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u = 0 \tag{55}$$

is a travelling wave solution of the form

$$u(x,t) = f(x - ct) = f(\xi) \tag{56}$$

where c is constant and $f(\xi) \to 0$ as $\xi \to \pm\infty$.

Discovery of Solitons

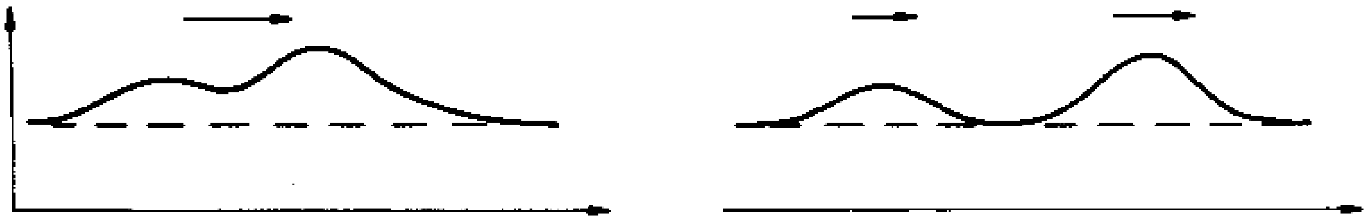
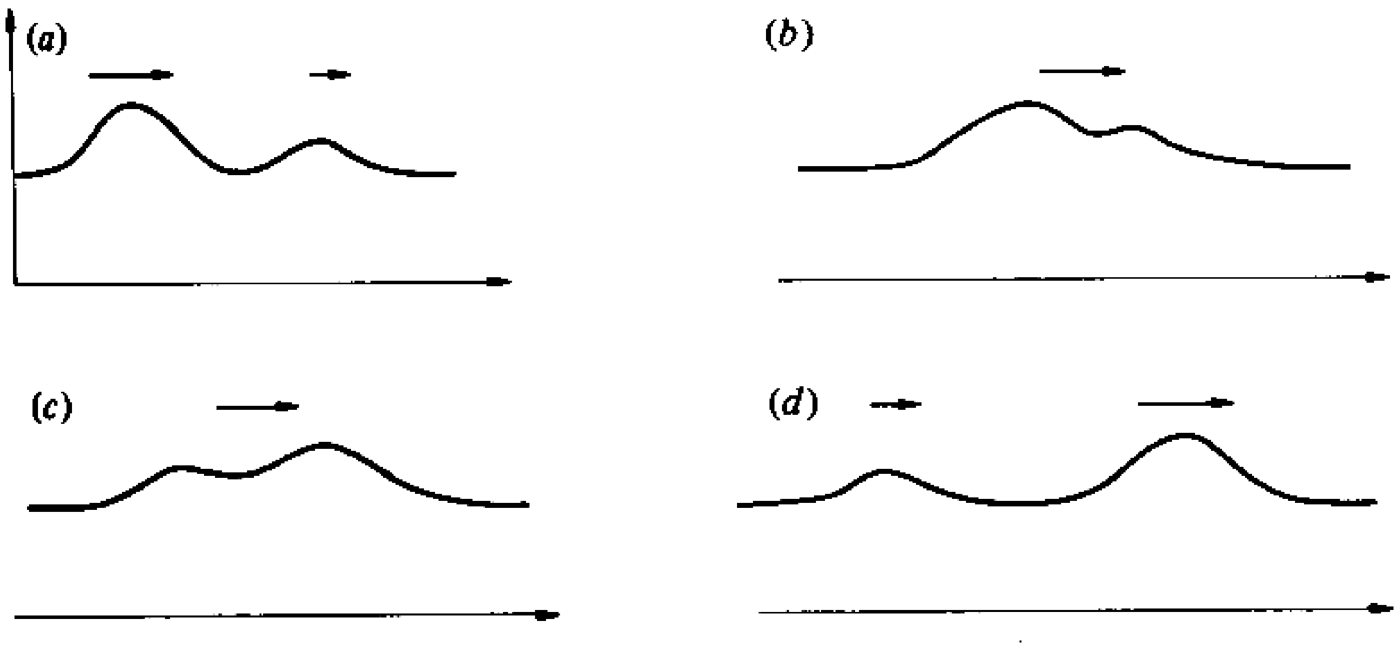


Fig. 1.6 A sketch depicting the interaction of two 'solitons', for times (*a*) $t = t_1$; (*b*) $t = t_2 > t_1$; (*c*) $t = t_3 > t_2$; (*d*) $t = t_4 > t_3$.



- Fermi, Pasta and Ulam (1955). Working on a numerical model of phonons in an anharmonic lattice. No equipartition of energy among the modes.

- Kruskal and Zabusky (1965).

  They considered the following equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0$$

They considered periodic boundary conditions

$$u(x,0) = \cos \pi x \quad \text{for } 0 < x \leq 2 \tag{57}$$

$$u(x+2,t) = u(x,t) \tag{58}$$

$$u_x(x+2,t) = u_x(x,t) \tag{59}$$

$$u_{xx}(x+2,t) = u_{xx}(x,t) \tag{60}$$

$$\delta = 0 \cdot 022. \tag{61}$$

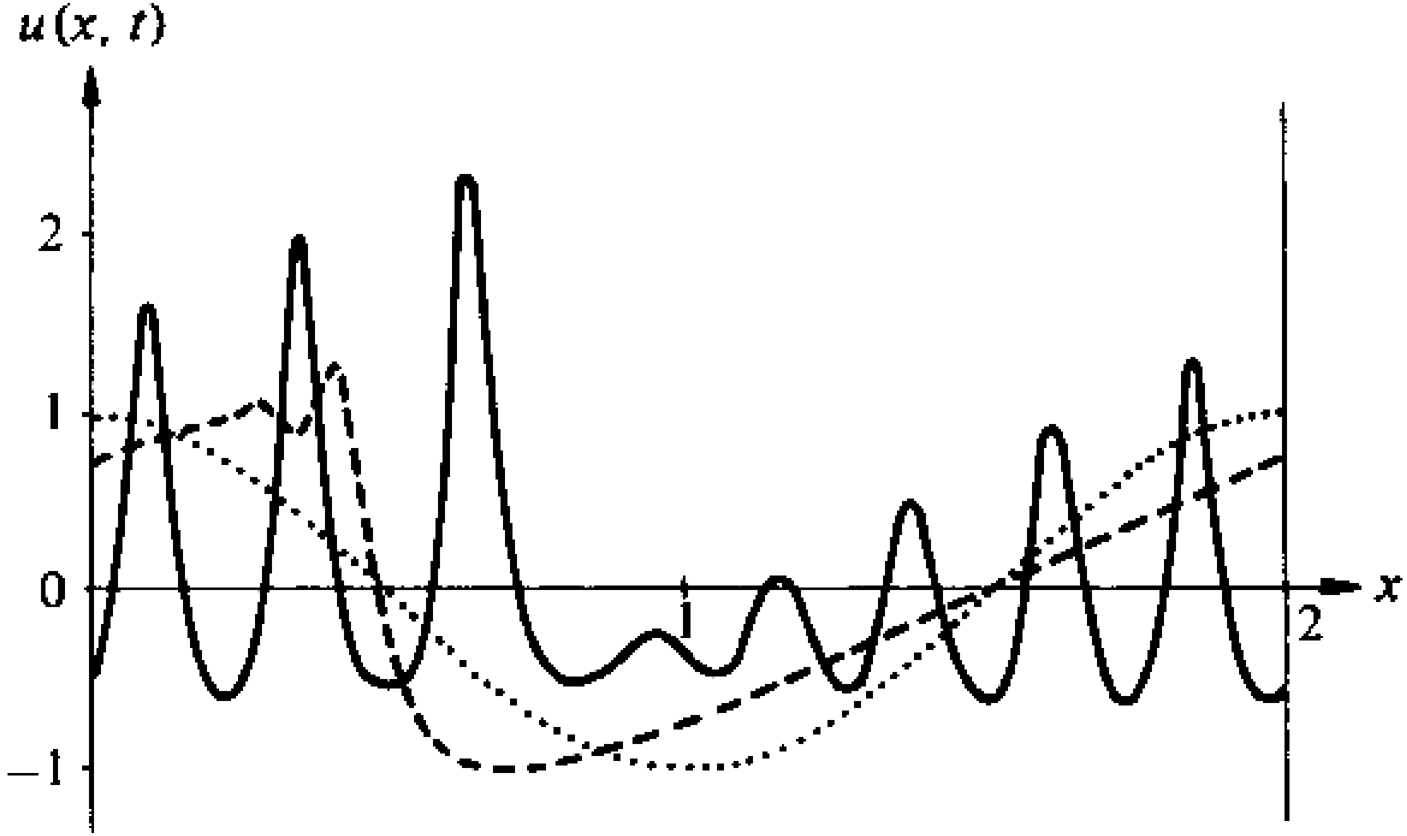


Figure 5: The solution of the periodic boundary-value problem for the KdV equation

Defining properties of solitons

- A nonlinear wave of permanent form.
- Localized.
- May interact strongly with other soliton and yet retain its identity.

Def.: *A* ***soliton*** *is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally with another (arbitrary) localized disturbance.*

Applications

Consider a linear wave motion with dispersion. The dispersion relation will have the form

$$\omega(k) = kc(k^2). \tag{62}$$

As $k \to 0$

$$\frac{\omega}{k} \sim c_0 - \lambda k^2, \quad \lambda > 0. \tag{63}$$

Such dispersion relation is obtained by the equation

$$u_t + c_0 u_x + \lambda u_{xxx} = 0$$

Wave propagation in a classical continuum $\Rightarrow$ the time evolution is given by the material derivative:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right). \tag{64}$$

The balance of nonlinearity and dispersion gives

$$u_t + c_0 u_x + \alpha(u u_x + \lambda u_{xxx}) = 0,\ \alpha \text{ small.}$$

Thus we have

$$u_\tau + u u_\xi + \lambda u_{\xi\xi\xi} = 0; \quad \xi = x - c_0 t, \quad \tau = \alpha t. \tag{65}$$

This KdV equation is valid if $x - c_0 t = O(1), t = O(\alpha^{-1})$ as $\alpha \to 0$.

Def.: *The KdV equation is the characteristic equation governing weekly nonlinear waves whose phase speed attains a simple maximum for wave of infinite length.*

- Long weekly nonlinear water waves;
- gravity waves in a stratified fluid;
- waves in a rotating atmosphere;
- ion-acustic fluid in a plasma;
- pressure waves in a liquid-gas bubble mixture.
- Examples of other nonlinear equations with a wide application: non linear Schrödinger equation and sine-Gordon equation.

2. Waves of permanent form

Travelling waves

Consider the nonlinear PDE

$$N\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u = 0. \tag{66}$$

Then ***guess*** that

$$u(x,t) = f(\xi), \quad \xi = x - ct$$

for some ***constant*** c.

NB Solution of this type ***do not always*** exist.

Solitary waves

Consider the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

We try a solution of the form

$$u(x,t) = f(\xi) \tag{67}$$

and we ask for

$$f, f', f'' \to 0, \text{ as } \xi \to \pm\infty$$

Substituting $f(\xi)$ in the KdV equation yields

$$-cf' - 6ff' + f''' = 0. \tag{68}$$

By integrating we have

$$-cf - 3f^2 + f'' = A \tag{69}$$

$$-cff' - 3f^2 f' + f'f'' = Af' \tag{70}$$

$$-\frac{1}{2}cf^2 - f^3 + \frac{1}{2}\left(f'\right)^2 = Af + B \tag{71}$$

The condition

$$f, f', f'' \to 0, \text{ as } \xi \to \pm\infty$$

implies

$$A = B = 0. \tag{72}$$

Therefore, we have

$$(f')^2 = f^2(2f + c)$$

$$\xi = \int \frac{d\xi}{df} df = \int \frac{df}{f'} = \pm \int \frac{df}{f\left(2f + c\right)^{1/2}} \tag{73}$$

Now we make the substitution

$$f = -\frac{1}{2} c \,\mathrm{sech}^2\, \Theta, \tag{74}$$

then we obtain

$$df = -2f \tanh \Theta d\Theta. \tag{75}$$

Finally, the integral becomes

$$\xi = \pm \int \frac{2f \tanh \Theta}{f\left(-c \operatorname{sech}^2 \Theta + c\right)^{1/2}} d\Theta = \pm \frac{2}{c^{1/2}} \Theta + \text{const. if } c > 0. \quad (76)$$

$$f(\xi) = -\frac{1}{2} c \operatorname{sech}^2 \left[\frac{1}{2} c^{1/2} (x - ct - x_0)\right] \qquad (78)$$

for aribatrary x_0 and $c \geq 0$.

General waves of permanent form

We have found that

$$\frac{1}{2}\left(f'\right)^2 = f^3 + \frac{1}{2}cf^2 + Af + B = F(f) \qquad (80)$$

Or equivalently

$$f' = \pm\sqrt{2F(f)}$$

Qualitative properties of f

- For real solution f, $F(f) \geq 0$;
- f' changes its sign only at a zero, f_1, of $F(f)$;
- $f(\xi)$ increase or decreases monotonically untill $F(f) = 0$.

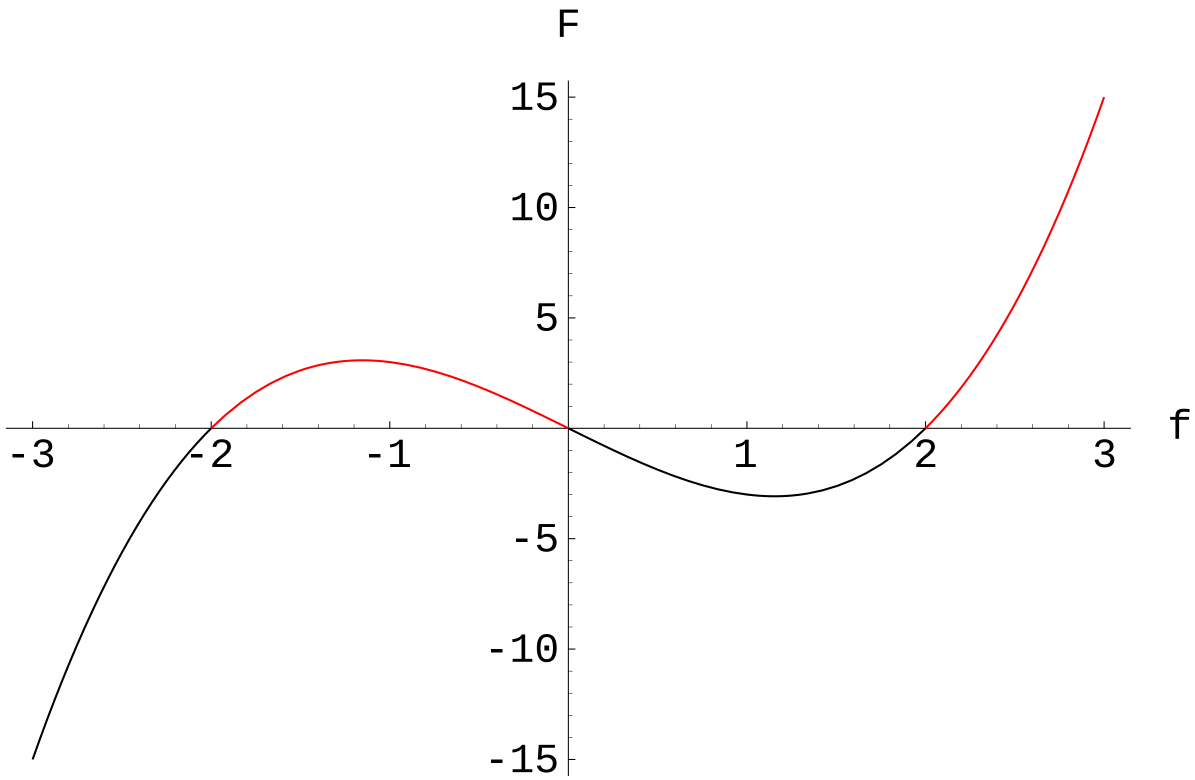


Figure 6: $F(f)$ with three simple zeros (cnoidal or periodic wave).

Behaviour of $f(\xi)$ in a neighbourhood of the zeros of $F(f)$

(i) If f_1 is a simple zero, then

$$(f')^2 = 2(f - f_1)F'(f_1) + O\left[(f - f_1)^2\right]. \qquad (81)$$

If we differentiate both sides w.r.t. ξ we get

$$2f'f'' = 2f'F'(f_1) + O[f'(f - f_1)] \qquad (82)$$

$$f'' = F'(f_1) + O[(f - f_1)]. \qquad (83)$$

Therefore $f''(\xi_1) = F'(f_1)$. Since $f'(\xi_1) = 0$

$$f(\xi) = f_1 + \frac{1}{2}(\xi - \xi_1)^2 F'(f_1) + O[(\xi - \xi_1)^3] \qquad (85)$$

(ii) If f_1 is a double zero, then

$$(f')^2 = (f - f_1)^2 F''(f_1) + O[(f - f_1)^3]. \tag{86}$$

This equation can be solved only if $F(f_1)'' > 0$. This time we obtain

$$f(\xi) - f_1 \sim \alpha \exp\left[\pm\xi\,(F''(f_1))^{1/2}\right] \quad \text{as } \xi \to \mp\infty \tag{88}$$

The solution extends from $-\infty$ to ∞ and can have only one peak.

(iii) If f_1 is a triple zero, then $f_1 = -c/6$, $A = 3(c/6)^2$ and $B = (c/6)^3$. Then we have

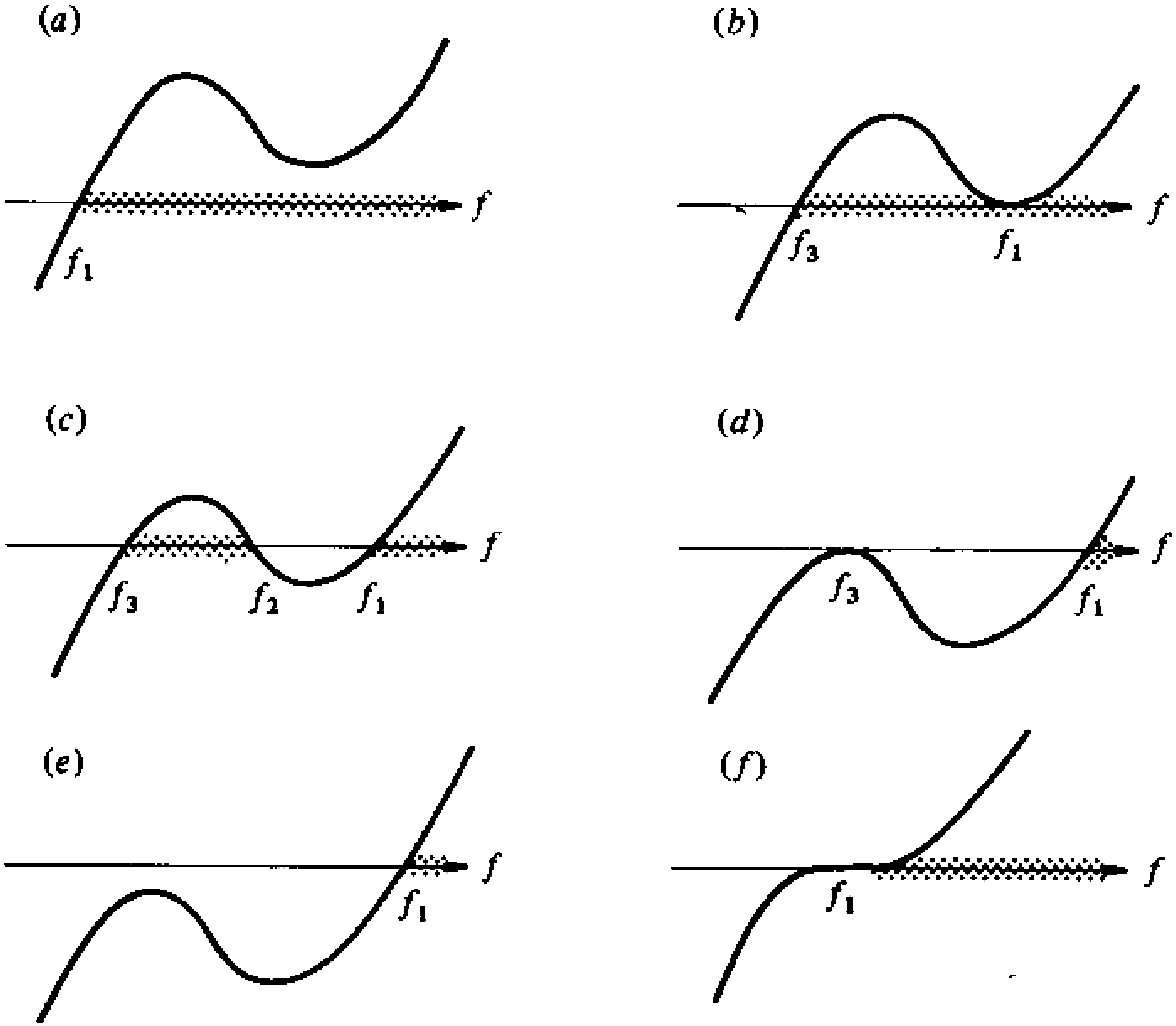
$$f' = \pm\sqrt{2}\left(f + \frac{c}{6}\right)^{3/2} \tag{89}$$

$$\xi = \pm\int \frac{df}{\sqrt{2}\left(f + \frac{c}{6}\right)^{3/2}}. \tag{90}$$

Finally, we obtain

$$f(\xi) = -\frac{c}{6} + \frac{2}{(\xi - \beta)}, \quad (92)$$

where β is a constant of integration. This solution al ways diverges at $\xi = \beta$.


(a)
f
f1
(b)
f
f3
f1
(c)
f
f3
f2
f1
(d)
f
f3
f1
(e)
f
f1
(f)
f
f1

- In the cases (a), (d), (e), (f) the solution is always unbounded.

- In case (b), $F(f)$ has a double zero at f_1 and a simple zero at f_3. $f(\xi)$ has a simple minimum at f_3 and attains its maximum f_1 exponentially as $\xi \to \pm\infty$. This is ***the solitary wave***.

- In case (c), $F(f)$ has simple zeros at f_3, f_2, so f has a simple maximum at f_2 and a simple minimum at f_3 with motion of period

$$\oint d\xi = 2\int_{f_2}^{f_3} \frac{d\xi}{df} df = 2\int_{f_2}^{f_3} \frac{df}{\sqrt{2F(f)}}. \tag{94}$$

These periodic solution are called ***cnoidal waves***, because they can be expressed in terms of the Jacobian of the elliptic function cn (when F is the cubic of the KdV equation).

Consider

$$f'' = -dV/df$$

for a given 'potential' function $V(f)$, where $f' = df/d\xi$. By writing

$$f'' = \frac{df'}{df} f' \qquad (95)$$

and integrating w.r.t. f the equation in the red box we obtain

$$\frac{1}{2}(f')^2 = E - V(f) = F(f) \qquad (97)$$

For some constant of integration (energy) E.

General features of $f(\xi)$

- There exist a periodic solution if $F(f)$ is positive between ***two simple zero***.

- There exist solitary-wave solutions if $F(f)$ is positive between ***a simple zero and a double zero*** of $F(f)$.

- If $F(f)$ is positive between ***two double zeros***, f_1 and f_2, then $f(\xi) \to f_1$ as $\xi \to \pm\infty$ and $f(\xi) \to f_2$ as $\xi \to \mp\infty$. These solutions are called ***kink*** or ***topological solitons*** (sine-Gordon equation).

Example

Consider the *sine-Gordon* equation

$$u_{tt} - u_{xx} + \sin u = 0$$

Look for a solution of the form $u = f(\xi)$, where $\xi = x - ct$ for some given constant c.

Substituting $f(\xi)$ in the sine-Gordon equation yields

$$c^2 f'' - f'' + \sin f = 0. \tag{98}$$

By using the identity $f'' = f' df'/df$ we obtain

$$(c^2 - 1)\frac{d\left[\frac{1}{2}(f')^2\right]}{df} + \sin f = 0. \tag{99}$$

Integration with respect to f gives

$$(c^2 - 1)\frac{1}{2}(f')^2 - \cos f = \text{const.} \tag{100}$$

$$(c^2 - 1)\frac{1}{2}(f')^2 - \left(1 - 2\sin^2\frac{f}{2}\right) = \text{const.} \tag{101}$$

$$(c^2 - 1)\frac{1}{2}(f')^2 + 2\sin^2\frac{f}{2} = A \tag{102}$$

For some arbitrary constant A.

In our previous notation we have

$$\frac{1}{2}(f')^2 = F(f) = \frac{2\sin^2\frac{1}{2}f - A}{1-c^2}. \tag{104}$$

Let us set $A = 0$

$F(f) > 0$ ***only if*** $0 < c^2 < 1$

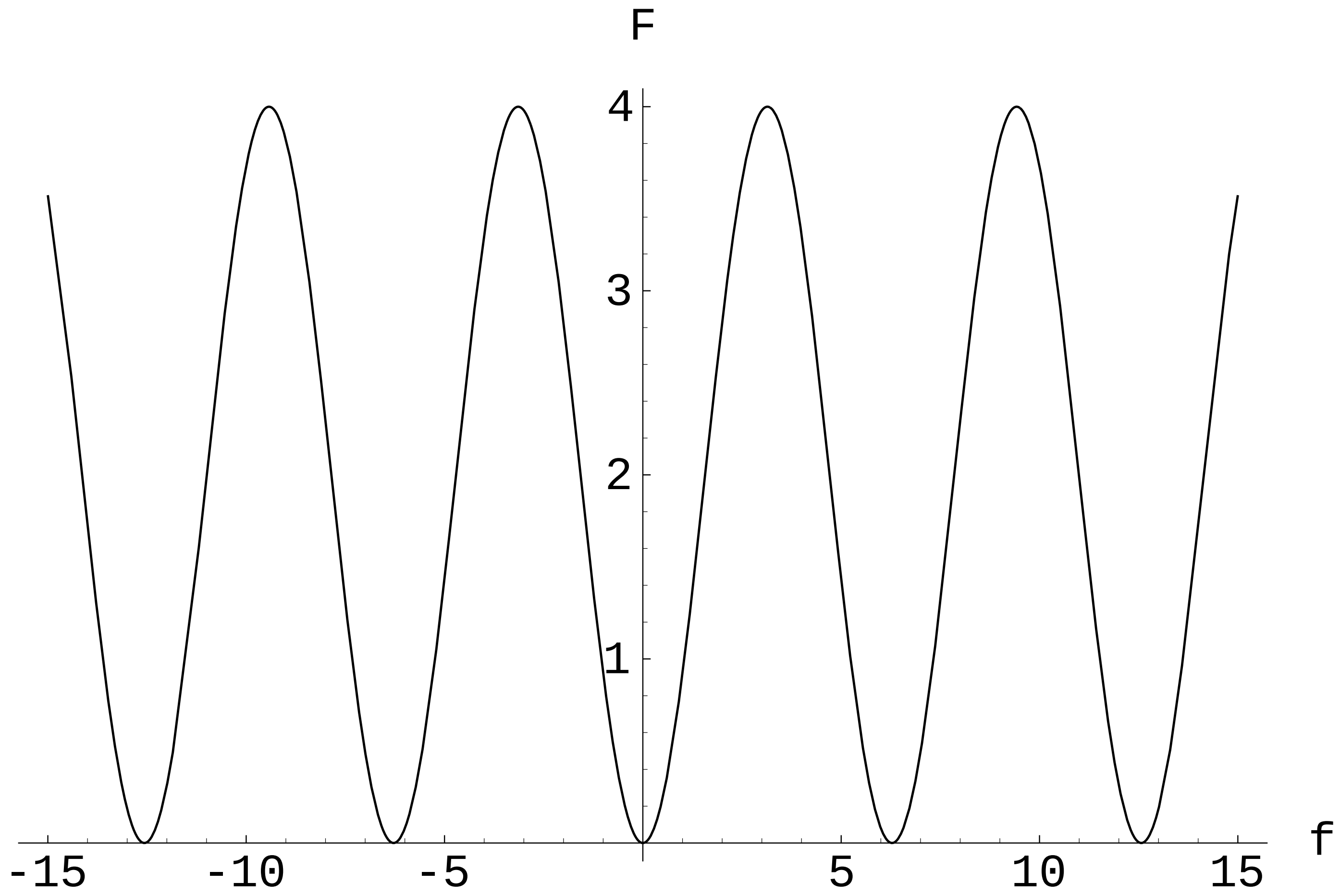


Figure 7: $F(f)$ against f for the sine-Gordon equation with $A = 0$ and $c^2 = 1/2$.

Now we have

$$F'(f) = \frac{2\sin\frac{f}{2}\cos\frac{f}{2}}{1-c^2} \tag{105}$$

$$F''(f) = \frac{\cos^2\frac{f}{2} - \sin^2\frac{f}{2}}{1-c^2} = \frac{\cos f}{1-c^2}. \tag{106}$$

$$F(f) = 0, \quad \text{at } f = 2\pi m, \quad m = \pm 1, \pm 2, \ldots \tag{110}$$

$$F'(2\pi m) = 0 \tag{111}$$

$$F''(2\pi m) = \frac{1}{1-c^2} \neq 0 \tag{112}$$

All the zeros are double. The solutions are kinks.

Now, let us make the substitution

$$v = \tan\left(\frac{1}{4}f\right). \tag{113}$$

Differentiating w.r.t. ξ yields

$$\begin{aligned} v' &= \frac{1}{4}f' \sec^2\left(\frac{1}{4}f\right) = \pm\frac{1}{2}\frac{\sin\left(\frac{1}{2}f\right)}{\sqrt{1-c^2}}\sec^2\left(\frac{1}{4}f\right) \\ &= \pm\frac{\sin\left(\frac{f}{4}\right)\cos\left(\frac{f}{4}\right)}{\sqrt{1-c^2}}\sec^2\left(\frac{1}{4}f\right) = \pm\frac{v}{\sqrt{1-c^2}}. \end{aligned} \tag{114}$$

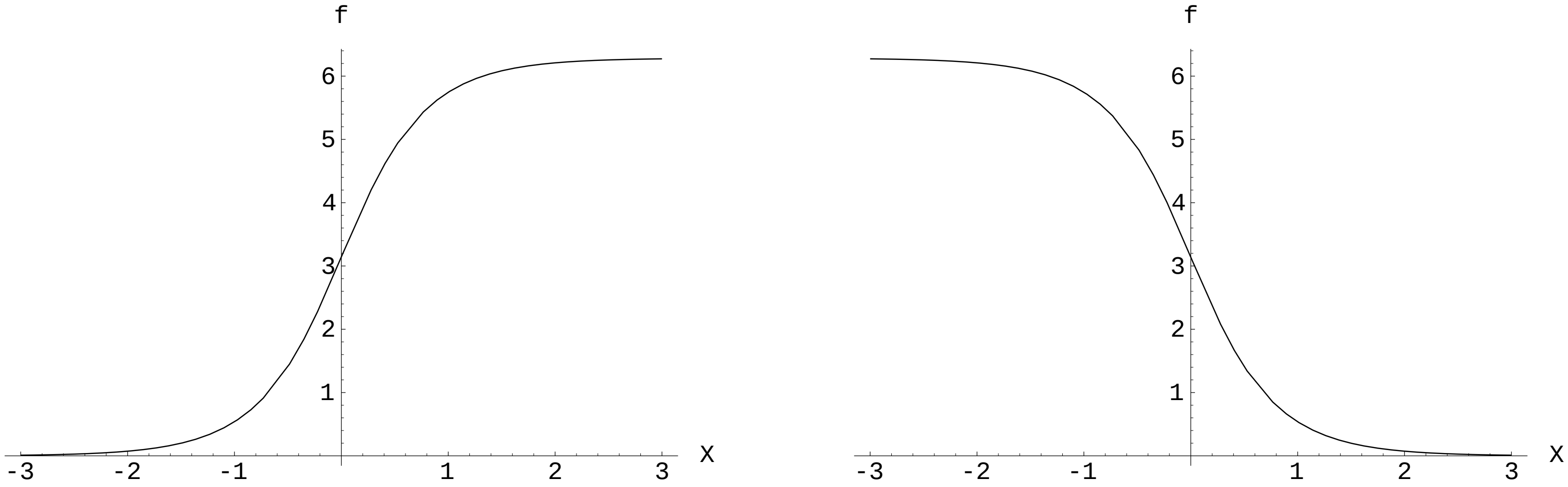
Finally, we have

$$f(\xi) = 4 \arctan \left\{ \pm \exp \left[\pm \frac{(\xi - \xi_0)}{\sqrt{1 - c^2}} \right] \right\} \qquad (117)$$

or

$$f(\xi) = 4 \arctan \left\{ \mp \exp \left[\pm \frac{(\xi - \xi_0)}{\sqrt{1 - c^2}} \right] \right\} \qquad (118)$$

If both signs are the same, we have a positive kink, if they differ we have a negative kink (antikink)



(a) A kink (b) An antikink

Figure 8: Kink and antikink of the sine-Gordon equation with $A = 0$ and $c^2 = 1/2$

Consider the following solution of the sine-Gordon equation:

$$\tan\left(\frac{1}{4}u\right) = \frac{c\sinh\left[\frac{x}{\sqrt{1-c^2}}\right]}{\cosh\left[\frac{ct}{\sqrt{1-c^2}}\right]}, \quad 0 < c^2 < 1 \tag{120}$$

Let us set $a = 1/\sqrt{1-c^2}$ and $v = \tan(f/4)$.

We have

$$v \sim c\frac{e^{ax} - e^{-ax}}{e^{-act}} \sim c\left(e^{a(x+ct)} - e^{-a(x-ct)}\right), \quad t \to -\infty. \tag{121}$$

This implies

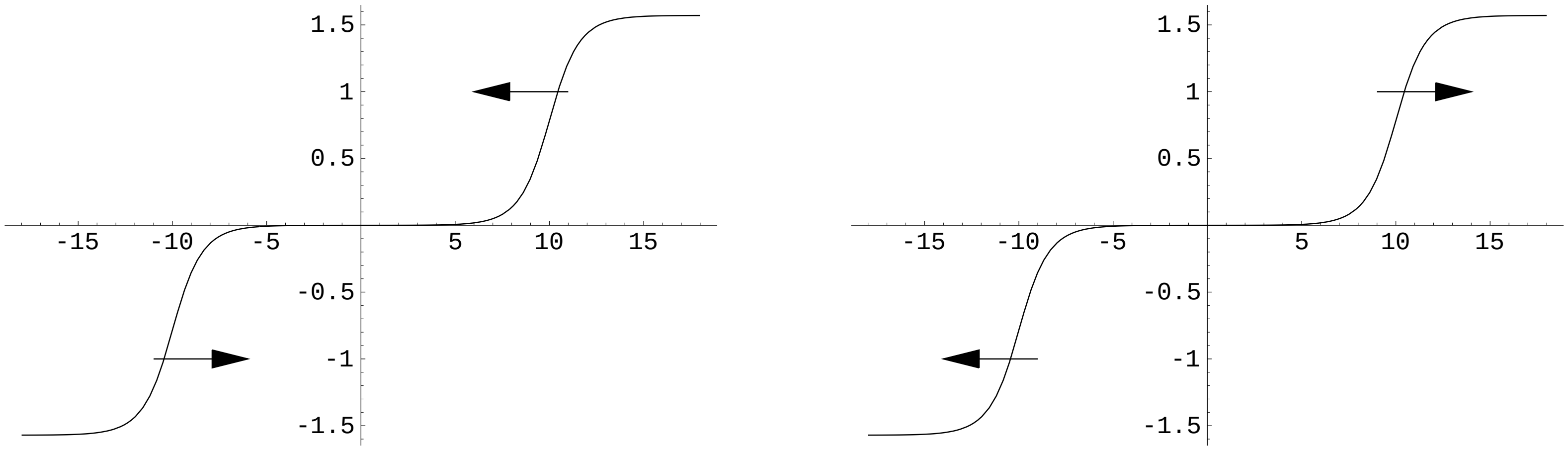
$$v \sim \begin{cases} ce^{a(x+ct)} & \text{if } x > -ct \\ ce^{a(x+ct)} - ce^{-a(x-ct)} & \text{if } ct < x < -ct \\ -ce^{-a(x-ct)} & \text{if } x < ct \end{cases}, \quad t \to -\infty. \tag{122}$$

Similarly, we have

$$v \sim c\frac{e^{ax} - e^{-ax}}{e^{act}} \sim c\left(e^{a(x-ct)} - e^{-a(x+ct)}\right), \quad t \to \infty. \tag{123}$$

This implies

$$v \sim \begin{cases} ce^{a(x-ct)} & \text{if } x > ct \\ ce^{a(x-ct)} - ce^{-a(x+ct)} & \text{if } -ct < x < ct \\ -ce^{-a(x+ct)} & \text{if } x < -ct \end{cases}, \quad t \to \infty. \tag{124}$$



(a) The solution (120) as $t \to -\infty$

(b) The solution (120) as $t \to \infty$

Figure 9: The solution (120) of the sine-Gordon equation as $t \to -\infty$ and as $t \to \infty$.

Example

The *Gardner* equation

$$u_t - 6uu_x + u_{xxx} = 12\delta u^2 u_x.$$

We look for a solution of form $u(x,t) = f(\xi)$, where $\xi = x - ct$. We then go through exactly the same procedure as for the KdV equation.

We have

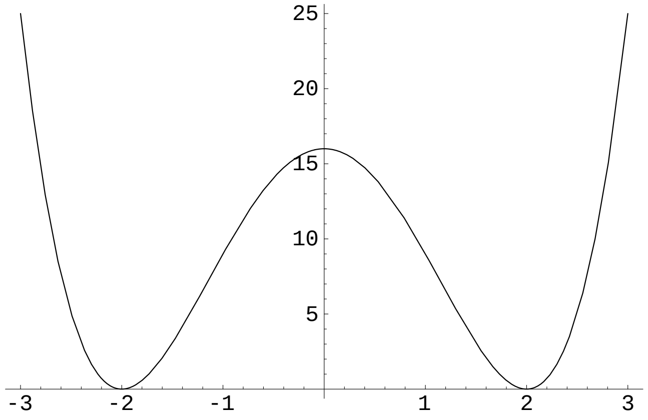
$$-cf' - 6ff' + f''' - 12\delta f^2 f' = 0 \tag{125}$$

$$-cf - 3f^2 + f'' - 4\delta f^3 = A \tag{126}$$

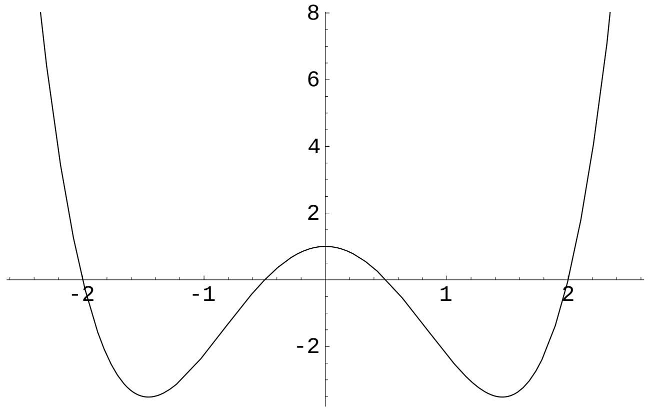
$$-cff' - 3f^2 f' + f'f'' - 4\delta f^3 f' = Af' \tag{127}$$

$$-\frac{1}{2}cf^2 - f^3 + \frac{1}{2}\left(f'\right)^2 + \delta f^4 = Af + B \tag{128}$$

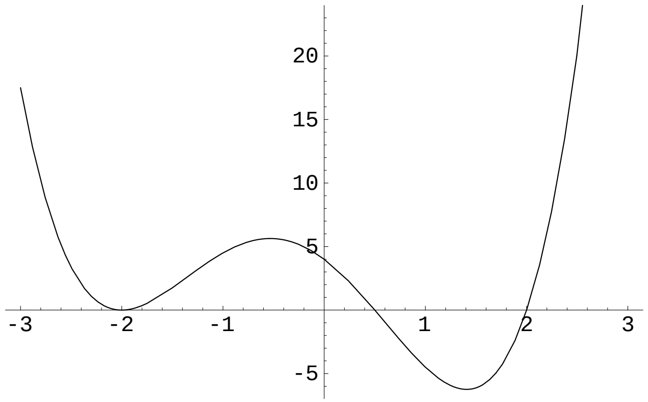
$$\frac{1}{2}\left(f'\right)^2 = \delta f^4 + f^3 + \frac{1}{2}cf^2 + Af + B = F(f) \tag{130}$$



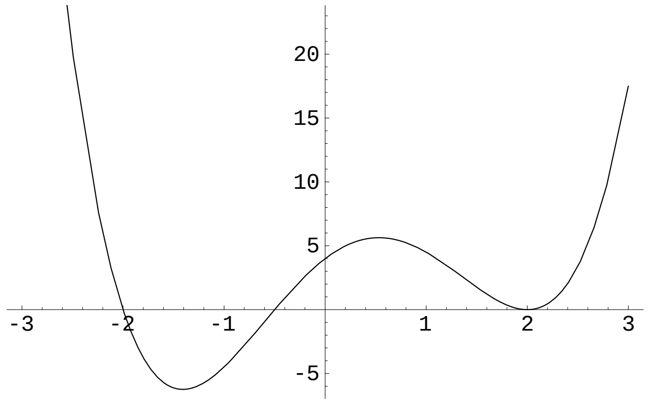
(a) One kink solution



(b) One periodic solution

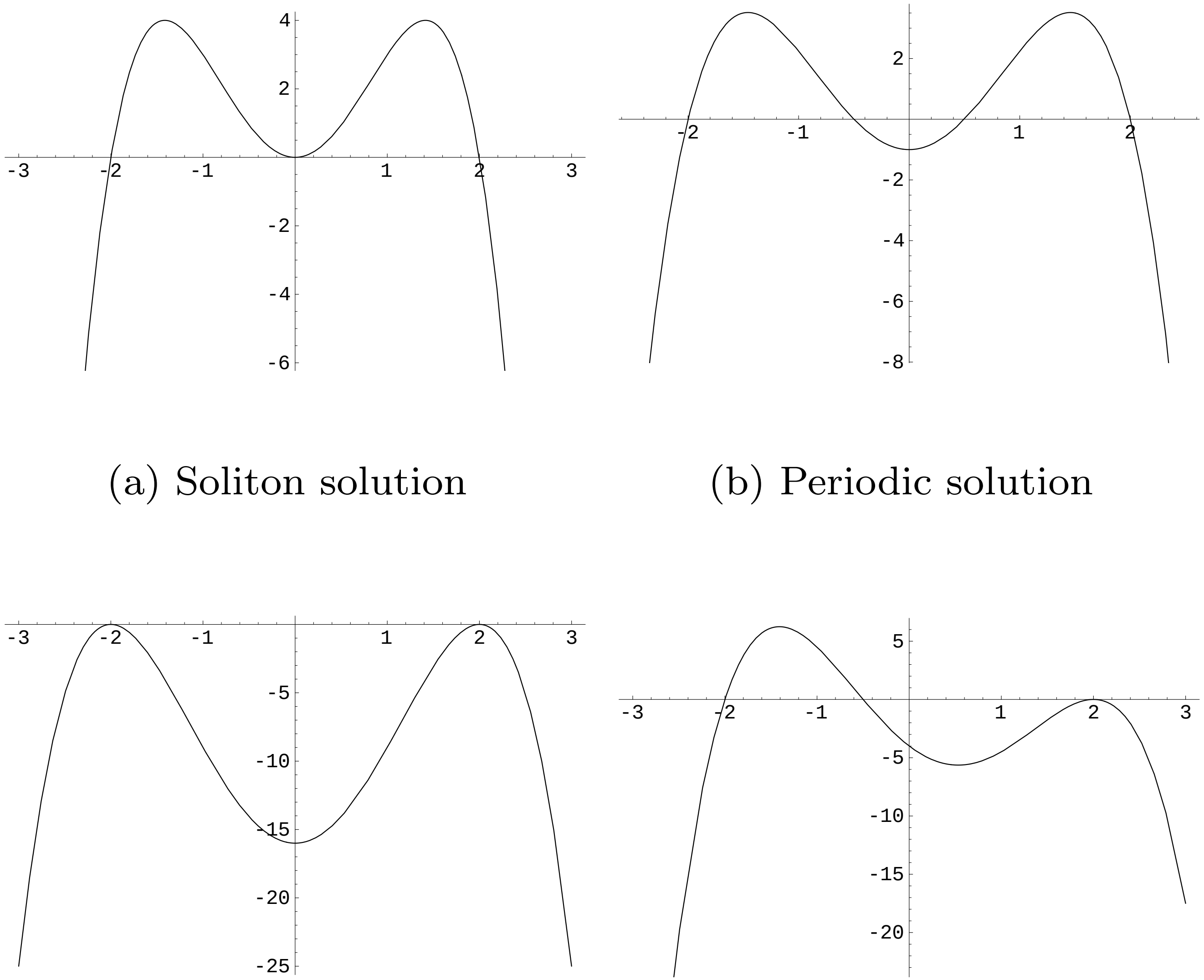


(c) Soliton solutions



(d) Soliton solution

Figure 10: Positive δ



(a) Soliton solution

(b) Periodic solution

(c) No physical solution

(d) Periodic solution

Figure 11: Negative δ

3.1 The Scattering Problem

Consider the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0.$$

We are looking for a general procedure to integrate this equation.

Let us introduce the ***Miura transformation***

$$u = v^2 + v_x$$

Direct substitution leads to

$$2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx}) \\ + 6v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 0, \quad (1)$$

which can be rearranged to give

$$\left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2 v_x + v_{xxx}) = 0. \quad (2)$$

The equation

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

is called ***modified KdV equation*** or ***mKdV***.

The Miura transformation

$$u = v^2 + v_x \tag{3}$$

is also known as ***Riccati equation*** for v, and can be linearized by the substitution $v = \psi_x / \psi$:

$$\psi_{xx} - u\psi = 0. \tag{5}$$

We now observe that the KdV equation is ***Galilean invariant***, *i.e.*

$$u \to \lambda + u(x + 6\lambda t, t), \quad -\infty < \lambda < \infty. \tag{6}$$

The equation for ψ now becomes

$$\psi_{xx} + (\lambda - u)\psi = 0, \quad -\infty < x < \infty \tag{8}$$

This is the ***time-independent Scrödinger equation.*** The eigenvalue problem defined by the parameter λ is the ***scattering problem*** (or *Sturm-Liouville* problem).

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{scattering}} & S(0) \\
{\scriptstyle \text{KdV}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{time evolution}} \\
u(x,t) & \xleftarrow[\text{scattering}]{\text{inverse}} & S(t)
\end{array}
$$

The diagram of the inverse scattering for the KdV equation.

Example

Consider the equation

$$u_t + u_x + u_{xxx} = 0.$$

Suppose we are given the initial-value problem $u(x, 0) = f(x)$.

Then

$$f(x) = \int_{-\infty}^{\infty} A(k) e^{ikx} dk \quad \text{and} \quad A(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad (9)$$

Here $A(k)$ plays the role of the 'scattering data'.

Now, the dispersion relation for the previous equation is

$$\omega(k) = k - k^3.$$

Therefore the solution to our linear PDE is

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega(k)t)} dk. \quad (11)$$

Use of the Fourier transform to solve linear PDE

1. Initial-value problem $u(x, 0) = f(x)$;
2. apply the Fourier transform to $f(x)$ to determine the 'scattering data' $A(k)$;
3. time evolution of the scattering data given by $A(k)e^{-i\omega(k)t}$;
4. reconstruct $u(x, t)$ by applying the inverse Fourier transform to $A(k)e^{-i\omega(k)t}$.

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{F.T.}} & A(k) \\
\text{PDE}\big\downarrow & & \big\downarrow\text{time evolution} \\
u(x,t) & \xleftarrow[\text{F.T.}]{\text{inverse}} & A(k)e^{-i\omega(k)t}
\end{array}
$$

Diagram of the use of the Fourier transform to solve linear PDEs

In order that appropriate solutions exist we shall require

$$\int_{-\infty}^{\infty} |u(x)| \, dx < \infty, \tag{12}$$

$$\int_{-\infty}^{\infty} (1 + |x|) \, |u(x)| \, dx < \infty \tag{13}$$

and

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx < \infty. \tag{14}$$

We shall also assume ψ and ψ_x to be continuous.

λ is called ***eigenvalue*** *and $\psi(x;\lambda)$ the relative* ***eigenfunction****.*

The operator

$$L = \frac{\partial^2}{\partial x^2} - u \tag{15}$$

is linear. The eigenvalue problem can be written $L\psi = -\lambda\psi.$

The function space $\mathcal{H}$, $\psi \in \mathcal{H}$ is an infinite dimensional linear space also called ***Hilbert space***. The scalar product (ψ, ϕ) of $\psi, \phi \in \mathcal{H}$ is defined by

$$(\psi, \phi) = \int_{-\infty}^{\infty} \psi(x)\phi^*(x)dx \tag{16}$$

Because u is integrable, $u \to 0$ as $x \to \pm\infty$. Therefore

$$\psi_{xx} \sim -\lambda\psi, \quad x \to \pm\infty \tag{17}$$

If λ is negative

$$\psi(x) \sim \begin{cases} \alpha e^{(-\lambda)^{1/2}x} & \text{as } x \to -\infty \\ \beta e^{-(-\lambda)^{1/2}x} & \text{as } x \to \infty. \end{cases} \tag{18}$$

The λ with this property are discrete. This constitute the ***discrete*** *spectrum.*

If λ is positive the 'eigenfunctions' are asymptotically a linear combination of

$$e^{\pm i\lambda^{1/2}x}.$$

This is the continuum spectrum.

The 'eigenfunctions' $e^{\pm i\lambda^{1/2}x}$ ***are not*** square-integrable!

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) e^{ikx} dk, \tag{19}$$

where $\lambda^{1/2} = k$.

- Discrete spectrum: $\kappa_n = (-\lambda_n)^{1/2}$ and $\kappa_1 < \kappa_2, \ldots < \kappa_N$. ψ_n is characterized by

$$\psi_n(x) \sim c_n \exp(-\kappa_n x), \quad x \to \infty. \tag{22}$$

Furthermore $\int_{-\infty}^{\infty} |\psi_n|^2 \, dx = 1$.

- Continuum spectrum. We shall consider eigenfunctions of the type

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + be^{ikx} & \text{as } x \to \infty \\ ae^{-ikx} & \text{as } x \to -\infty. \end{cases} \tag{23}$$

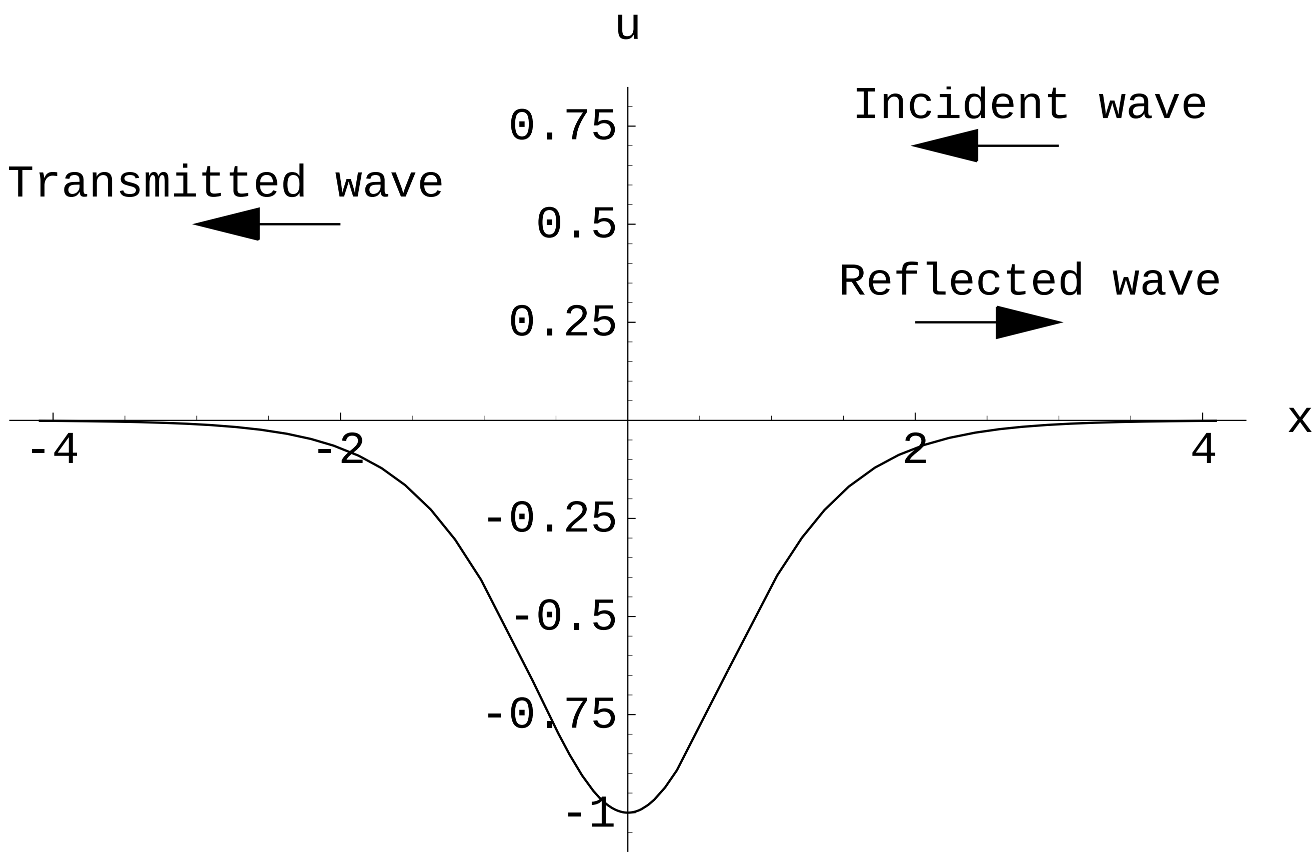


Figure 1: Schematic representation of incident, reflected and transmitted wave

It can be shown that

- If $u(x) \geq 0$, $-\infty < x < \infty$ there is no discrete spectrum;
- if $u(x) \leq 0$, $-\infty < x < \infty$ and $u(x) \to 0$ 'sufficiently rapidly', then there is only a finite number of discrete eigenvalues;
- $|a|^2 + |b|^2 = 1$ (in quantum mechanics this is the conservation of probability).

Consider two different discrete eigenfunctions (for the same u):

$$\psi_m'' - (\kappa_m^2 + u)\psi_m = 0, \quad \psi_n'' - (\kappa_n^2 + u)\psi_n = 0. \tag{24}$$

Therefore, we have

$$(\kappa_n^2 - \kappa_m^2)\psi_n\psi_m = \psi_m\psi_n'' - \psi_n\psi_m'' = \frac{d}{dx}W(\psi_m, \psi_n), \tag{25}$$

where $W(\psi_m, \psi_n)$ is the ***Wronskian*** of ψ_m, ψ_n. By integrating we have

$$[W(\psi_m, \psi_n)]_{-\infty}^{\infty} = (\kappa_n^2 - \kappa_m^2)\int_{-\infty}^{\infty} \psi_m\psi_n dx. \tag{26}$$

This implies $\int_{-\infty}^{\infty} \psi_m\psi_n dx = 0.$ That is, ψ_m and ψ_n are ***orthogonal***.

Example
Wave in an inhomogeneous medium

One-dimensional propagation of sound or light in an inhomogeneous medium is governed by

$$\nabla^2 \phi - \frac{1}{c^2(x)} \frac{\partial^2 \phi}{\partial t^2} = 0. \qquad (28)$$

We then insert in the above equation a normal mode

$$\phi(\mathbf{x}, t) = \psi(x) e^{i(ly + mz - \omega t)}. \qquad (29)$$

This yields to

$$\frac{d^2\psi}{dx^2} - (l^2 + m^2)\psi(x) = -\frac{\omega^2}{c^2}\psi(x). \tag{30}$$

We then define

$$\lambda = -\left(l^2 + m^2\right), \quad u(x) = -\frac{\omega^2}{c^2(x)}. \tag{31}$$

Therefore (30) becomes

$$\psi'' + (\lambda - u(x))\,\psi(x) = 0. \tag{32}$$

If now

$$c(x) \to c_\infty \quad \text{as } x \to \infty, \tag{33}$$

we coud redefine

$$\lambda = \frac{\omega^2}{c_\infty^2} - \left(l^2 + m^2\right), \quad u(x) = \omega^2 \left(\frac{1}{c_\infty^2} - \frac{1}{c^2(x)} \right). \tag{34}$$

Then we have that

$$u(x) \to 0 \quad \text{as } x \to \infty. \tag{35}$$

Example
The δ function potential

Consider

$$u(x) = -V\delta(x),$$

where $\delta(x)$ is Dirac's delta function.

Integrating once the Sturm-Liouville equation

$$\psi_{xx} + (\lambda - u)\psi = 0, \quad -\infty < x < \infty \tag{36}$$

yields

$$\begin{aligned}\psi'(\epsilon) - \psi'(-\epsilon) &= -\int_{-\epsilon}^{\epsilon} \left(V\delta(x) + \lambda\right)\psi(x)dx \\ &= -V\psi(0) - \lambda\int_{-\epsilon}^{\epsilon} \psi(x)dx.\end{aligned} \tag{37}$$

Finally, as $\epsilon \longrightarrow 0$ we have

$$\lim_{\epsilon \longrightarrow 0} \left(\psi'(\epsilon) - \psi'(-\epsilon)\right) = -V\psi(0). \tag{39}$$

ψ' is discontinuous at the origin.

For $x > 0$ and $x < 0$, ψ obeys the free particle equation

$$\psi'' + \lambda\psi = 0. \tag{40}$$

Since the eigenfunctions must be square-integrable, we have

$$\psi_n(x) = \begin{cases} c_n \exp\left(-\kappa_n x\right) & \text{if } x \geq 0, \\ d_n \exp\left(\kappa_n x\right) & \text{if } x < 0. \end{cases} \tag{42}$$

By continuity at $x = 0$, $c_n = d_n$. Then, the normalization condition leads to

$$\int_{-\infty}^{0} |c_n|^2 \exp(2\kappa_n x)dx + \int_{0}^{\infty} |c_n|^2 \exp(-2\kappa_n x)dx =$$

$$\frac{|c_n|^2}{\kappa_n} = 1.$$

Therefore, up to an arbitrary phase, we have

$$c_n = \sqrt{\kappa_n} \qquad (44)$$

We can now find the only discrete eigenvalue:

$$\lim_{\epsilon \longrightarrow 0} (\psi'(\epsilon) - \psi'(-\epsilon)) = -2c_n \kappa_n \lim_{\epsilon \longrightarrow 0} \exp(-\kappa_n \epsilon) = -V c_n, \qquad (45)$$

$c_n = \sqrt{\kappa_n}$.

Finally, we have

$$\kappa_n = \frac{V}{2} \qquad (47)$$

or equivalently $\lambda_1 = -V^2/4$.

Continuum spectrum

We have that $\lambda > 0$. Consider a wave of unit amplitude incident from the left:

$$\psi(x) = \begin{cases} e^{-ikx} + be^{ikx} & \text{if } x > 0 \\ ae^{-ikx} & \text{if } x < 0 \end{cases} \quad \text{as } x \to \infty, \tag{49}$$

Since ψ is continuous, we have

$$a = 1 + b \tag{50}$$

Using again

$$\lim_{\epsilon \longrightarrow 0} \left(\psi'(\epsilon) - \psi'(-\epsilon) \right) \tag{51}$$

yields

$$-ik + ibk - (-aik) = -V(1+b). \tag{52}$$

Finally, we obtain

$$b(k) = -\frac{V}{V+2ik}. \tag{54}$$

The transmission coefficient a is then equal to

$$a(k) = 1 + b(k) = \frac{2ik}{V + 2ik}. \tag{55}$$

Note that

$$|b(k)|^2 + |a(k)|^2 = \frac{V^2}{V^2 + 4k^2} + \frac{4k^2}{V^2 + 4k^2} = 1. \tag{57}$$

Example: $u(x) = -2\,\mathrm{sech}^2\, x$.

If $u(x) = -2\,\mathrm{sech}^2\, x$ then it may be verified that

$$\lambda_1 = -1, \quad \psi_1(x) \propto \mathrm{sech}\, x \tag{59}$$

This is the only eigenfunction (N=1).

To normalize, let us set $\psi_1(x) = a \operatorname{sech} x$:

$$\int_{-\infty}^{\infty} |a|^2 \operatorname{sech}^2 x dx = |a|^2 [\tanh x]_{-\infty}^{\infty} = 2 |a|^2 . \qquad (60)$$

Therefore, up to a phase factor, we have $a = 1/\sqrt{2}$ It can also be shown that

$$a(k) = \frac{ik-1}{ik+1}, \quad b(k) = 0 \quad \forall k \qquad (62)$$

$u(x) = -2 \operatorname{sech}^2 x$ is a ***reflectionless*** potential.

Example: $u(x) = -6\,\mathrm{sech}^2\, x.$

If $u(x) = -6\,\mathrm{sech}^2\, x$, it can be shown that $N = 2$ and

$$\lambda_1 = -1, \qquad \psi_1(x) = \sqrt{\frac{3}{2}}\tanh x\,\mathrm{sech}\, x \tag{66}$$

$$\lambda_2 = -4, \qquad \psi_2(x) = \frac{\sqrt{3}}{2}\,\mathrm{sech}^2\, x, \tag{67}$$

$$b(k) = 0 \qquad \forall k \tag{68}$$

Example: $u(x) = -V \operatorname{sech}^2 x$

The Sturm-Liouville equation is now

$$\psi'' + (\lambda + V \operatorname{sech}^2 x)\psi = 0 \qquad (70)$$

In order to solve this equation we make the substitution

$$y = \tanh x, \; -1 < y < 1 \text{ for } -\infty < x < \infty.$$

Therefore, we have

$$\frac{d}{dx} = \operatorname{sech}^2 x \frac{d}{dy} = (1 - y^2)\frac{d}{dy} \tag{71}$$

and so

$$(1 - y^2)\frac{d}{dy}\left[\left(1 - y^2\right)\frac{d}{dy}\right] + \left[\lambda + V\left(1 - y^2\right)\right]\psi = 0. \tag{72}$$

or

$$\frac{d}{dy}\left[\left(1 - y^2\right)\frac{d}{dy}\right] + \left[V + \frac{\lambda}{(1 - y^2)}\right]\psi = 0. \tag{74}$$

This is the ***associated Legendre equation***.

Discrete spectrum

First, suppose that $V = N(N+1)\ .$

If $\lambda = -\kappa^2 (< 0)$, *the bound solutions occur when*

$$\kappa_n = n, \quad n = 1, 2, \ldots N. \tag{75}$$

The eigenfunctions are proportional to the ***associated Legendre functions*** $P_N^n(y)$, *where*

$$P_N^n(y) = (-1)^n \left(1 - y^2\right)^{n/2} \frac{d^n}{dy^n} P_N(y) \tag{76}$$

and

$$P_N(y) = \frac{(-1)^N}{N! 2^N} \frac{d^N}{dy^N} \left(1 - y^2\right)^N , \tag{77}$$

$P_N(y)$ being the *Legendre polynomial* of degree N.

Continuum spectrum

If $\lambda = k^2 (> 0)$ *we look for solutions which behaves like*

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \to \infty \\ a(k)e^{-ikx} & \text{as } x \to -\infty. \end{cases} \tag{78}$$

For $x \to -\infty$ these are given by

$$\psi(x;k) = a(k)2^{ik}(\operatorname{sech} x)^{-ik} F(\tilde{a}, \tilde{b}; \tilde{c}; (1+y)/2). \tag{80}$$

$F(\alpha, \beta, ; \gamma; z)$ is the *hypergeometric function*, which is defined by the series

$$F(\alpha, \beta; \gamma; z) = 1+$$
$$\sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)n!} z^n \quad (82)$$

In our case

$$\tilde{a} = \frac{1}{2} - ik + \left(V + \frac{1}{4}\right) \quad (83)$$

$$\tilde{b} = \frac{1}{2} - ik - \left(V + \frac{1}{4}\right) \quad (84)$$

$$\tilde{c} = 1 - ik \quad (85)$$

It is fairly easy to show that

$$\psi(x;k) \sim a(k)e^{-ikx} \quad \text{as } x \to -\infty. \tag{86}$$

It can also be shown that

$$\begin{aligned}\psi(x;k) \sim \; & \frac{a\Gamma(\tilde{c})\Gamma(\tilde{a}+\tilde{b}-\tilde{c})}{\Gamma(\tilde{a})\Gamma(\tilde{b})}e^{-ikx} \\ & + \frac{a\Gamma(\tilde{c})\Gamma(\tilde{c}-\tilde{a}-\tilde{b})}{\Gamma(\tilde{c}-\tilde{a})\Gamma(\tilde{c}-\tilde{b})}e^{ikx} \quad \text{as } x \to \infty.\end{aligned} \tag{87}$$

Comparing the previous expressions with (78) yields

$$a(k) = \frac{\Gamma(\tilde{a})\Gamma(\tilde{b})}{\Gamma(\tilde{c})\Gamma(\tilde{a}+\tilde{b}-\tilde{c})} \quad \text{and} \quad b(k) = \frac{a(k)\Gamma(\tilde{c})\Gamma(\tilde{c}-\tilde{a}-\tilde{b})}{\Gamma(\tilde{c}-\tilde{a})\Gamma(\tilde{c}-\tilde{b})}$$

We now want to show that $b(k) = 0$, *i.e.* this potential is ***reflectionless.***

We need the identity

$$\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{1}{2}+z\right)=\frac{\pi}{\cos \pi z}. \tag{88}$$

Hence observe that

$$\Gamma(\tilde{c}-\tilde{a})\Gamma(\tilde{c}-\tilde{b}) \tag{89}$$

$$=\Gamma\left[\frac{1}{2}-\left(V+\frac{1}{4}\right)^{1/2}\right]\Gamma\left[\frac{1}{2}+\left(V+\frac{1}{4}\right)^{1/2}\right] \tag{90}$$

$$=\pi/\cos\left[\pi\left(V+\frac{1}{4}\right)^{1/2}\right] \tag{91}$$

It follows that $b(k) = 0$ if

$$\left(V + \frac{1}{4}\right)^{1/2} = N + \frac{1}{2} \tag{92}$$

or equivalently

$$V = N(N+1). \tag{94}$$

The case $V \neq N(N+1)$

The two coefficients $a(k)$ and $b(k)$ have poles where $\Gamma(\tilde{a})$ and $\Gamma(\tilde{b})$ have poles.

This happens at $\tilde{b} = -m, \; m = 0, 1, \ldots$ or

$$k = i \left[\left(V + \frac{1}{4} \right)^{1/2} - \left(m + \frac{1}{2} \right) \right]. \qquad (96)$$

There is a finite number of discrete eigenvalues if

$$\left(V+\frac{1}{4}\right)^{1/2} > \frac{1}{2} \quad i.e.\ V > 0. \tag{98}$$

Their number is

$$\left[\left(V+\frac{1}{4}\right)^{1/2} - \frac{1}{2}\right] + 1 \tag{100}$$

where $[z]$ denotes the integral part of z (the greatest integer $\leq z$).

The eigenvalues are given by

$$\kappa_m = \left(V + \frac{1}{4}\right)^{1/2} - \left(m + \frac{1}{2}\right) = \mu \quad (102)$$

The eigenfunctions are the ***associated Legendre functions*** $P_\nu^\mu(y)$, where ν is a solution of the equation

$$V = \nu(\nu + 1). \qquad (103)$$

3.2 The Inverse Scattering Problem

We have seen how from the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{104}$$

we can get to the Schrödinger equation

$$\psi_{xx} + (\lambda - u)\psi = 0, \quad -\infty < x < \infty \tag{105}$$

via the Miura transformation

$$u = v^2 + v_x \tag{106}$$

and the substitution $v = \psi_x/\psi$

Suppose that we are given $u(x)$ and

$$\psi'' + (\lambda - u)\psi = 0, \quad -\infty < x < \infty \tag{107}$$

The ***direct*** scattering problem is to deduce the scattering data, *i.e.*
The eigenfunctions and eigenvalues

$$\lambda_m = -\kappa_m^2, \quad \psi_m(x) \quad m = 1, 2, \ldots, N \tag{108}$$

transmission and reflection coefficients

$$a(k) \quad \text{and} \quad b(k) \quad \forall \lambda = k^2. \tag{109}$$

*The **inverse** scattering problem is to deduce $u(x)$ from the scattering data .*

The Marchenko equation

Gelfand & Levitan (1951) solved the inverse scattering problem by use of Fourier transform (with deep and difficult arguments). Their solution was simplified by Marchenko.

Let us consider the wave equation

$$\phi_{xx} - \phi_{zz} = 0. \qquad (111)$$

We then express $\phi(x,z)$ in term of its Fourier transform:

$$\phi(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(x;k)e^{-ikz}dk \qquad (112)$$

and

$$\psi(x;k) = \int_{-\infty}^{\infty} \phi(x,z)e^{ikz}dz \qquad (113)$$

Substituting the above integral into the wave equation yields

$$\psi_{xx} + k^2\psi = 0. \qquad (115)$$

Further, let us suppose that we are interested in a solution ψ of the previous equation such that

$$\psi \sim e^{ikx}, \quad x \to \infty. \quad (117)$$

This is obtained by setting

$$\phi(x, z) = \delta(x - z) + K(x, z), \qquad (118)$$

where $K(x, z) = 0$ if $z < x$, and obeys the classical wave equation.

By taking the Fourier transform of (118) we have

$$\psi(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz \quad (120)$$

The above wavefunction has the correct asymptotic value.

NB Note that

$$\int_{-\infty}^{\infty} \delta(x-z)e^{ikz} = e^{ikx} \quad \text{and} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-z)} = \delta(x-z) \quad (121)$$

Which equation will $K(x,z)$ *if we slightly modify* (115) *to*

$$\psi'' + (k^2 - u)\psi = 0? \tag{123}$$

The boundary conditions are always

$$\psi(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz \tag{124}$$

$K(x,z) = 0$ if $z < x$ and

$$\psi \sim e^{ikx}, \quad x \to \infty. \tag{125}$$

It can be shown that

$$K_{xx}(x,z) - K_{zz}(x,z) - u(x)K(x,z) = 0 \quad \text{for } z > x \tag{126}$$

and

$$u(x) = -2\frac{d\hat{K}}{dx} = -2\left\{K_x(x,x) + K_z(x,x)\right\} \tag{128}$$

with the condition

$$K(x,z), K_z(x,z) \to 0 \quad \text{as } z \to \infty. \tag{129}$$

Here $\hat{K}(x) = K(x,x)$.

The equation which one then tries to invert is

$$\hat{\psi} = \psi^* + b(k)\psi. \qquad (131)$$

with

$$\psi(x;k) = e^{ikx} + \int_x^\infty K(x,z)e^{ikz}dz. \qquad (132)$$

$\hat{\psi}$ has the right asymptotic limit:

$$\hat{\psi}(x;k) \sim e^{-ikx} + b(k)e^{ikx} \quad \text{as } x \to \infty. \qquad (133)$$

$K(x,z)$ can be found by solving the integral equation

$$K(x,z)+F(x+z)+\int_x^\infty K(x,y)F(y+z)dy = 0 \quad z > x > -\infty, \tag{135}$$

where $K(x,z) = 0$ if $z < x$.

The above equation is the ***Marchenko equation***, is a ***linear Fredholm integral equation.***

The function $F(X)$ is defined by

$$F(X) = \sum_{n=1}^{N} c_n^2 \exp\left(-\kappa_n X\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k) e^{ikX} dk \qquad (137)$$

where $b(k)$ is the reflection coefficient, $\kappa_n^2 = -\lambda_n$ and c_n are the normalization coefficients

$$\psi_n(x) \sim c_n \exp(-\kappa_n x), \quad x \to \infty \qquad (138)$$

or

$$c_n = \lim_{x \to \infty} \left[\psi_n(x) \exp(\kappa_n x)\right] \qquad (139)$$

Inverse scattering problem: summary

Consider the Scrödinger equation

$$\psi'' + (\lambda - u(x))\psi = 0. \tag{140}$$

Suppose we want to find $u(x)$ and are given the scattering data

$$\psi_n(x), \quad \lambda_n = -\kappa_n^2, \quad n = 1, \ldots, N \tag{142}$$

for the discrete spectrum and the reflection coefficient $b(k)$

for the continuum spectrum.

$$u(x) = -2\frac{d\hat{K}}{dx} = -2\left\{K_x(x,x) + K_z(x,x)\right\} \qquad (144)$$

with $K(x,z), K_z(x,z) \to 0 \quad$ as $z \to \infty$ and $K(x,z) = 0$ if $z < x$. Moreover $K(x,z)$ satisfies the Marchenko equation

$$K(x,z) + F(x+z) + \int_x^\infty K(x,y)F(y+z)dy = 0 \quad z > x > -\infty, \qquad (146)$$

Furthermore,

$$F(X) = \sum_{n=1}^{N} c_n^2 \exp\left(-\kappa_n X\right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k) e^{ikX} dk \qquad (148)$$

where c_n are the coefficients

$$\psi_n(x) \sim c_n \exp(-\kappa_n x), \quad x \to \infty \qquad (149)$$

or

$$c_n = \lim_{x \to \infty} \left[\psi_n(x) \exp(\kappa_n x)\right]. \qquad (150)$$

Solution of the Marchenko equation

It can be solved by iteration:

$$K_1(x,z) = \begin{cases} -F(x+z) & \text{if } z > x \\ 0 & \text{if } z < x \end{cases} \tag{151}$$

and

$$K_2(x,z) = -F(x+z) + \int_{-\infty}^{\infty} K_1(x,y)F(y+z)dy \tag{152}$$

$$K_3(x,z) = -F(x+z) + \int_{-\infty}^{\infty} K_2(x,y)F(y+z)dy. \tag{153}$$

It can be shown that $K_n(x,z) \to K(x,z)$.

This is the ***Neumann series***

Now, suppose that $F(x+z)$ is a ***separable*** function, *i.e.*

$$F(x+z) = \sum_{n=1}^{N} X_n(x) Z_n(z). \quad (155)$$

The Marchenko equation can therefore be written as

$$K(x,z) + \sum_{n=1}^{N} X_n(x) Z_n(z) + \sum_{n=1}^{N} Z_n(z) \int_x^{\infty} K(x,y) X_n(y) dy = 0 \quad (157)$$

The solution therefore must take the form

$$K(x,z) = \sum_{n=1}^{N} L_n(x) Z_n(z). \quad (159)$$

Upon this substitution for $K(x,z)$ we obtain

$$\sum_{n=1}^{N} L_n(x)Z_n(z) + \sum_{n=1}^{N} X_n(x)Z_n(z)$$

$$+\sum_{n=1}^{N} Z_n(z) \sum_{m=1}^{N} L_m(x) \int_{-\infty}^{\infty} Z_m(y)X_n(y)dy = 0. \quad (160)$$

In order for the equation to be identically zero in the variables x and z, each term in the external sum must be identically zero.

$$L_n(x) + X_n(x) + \sum_{m=1}^{N} L_m(x) \int_{-\infty}^{\infty} Z_m(y)X_n(y)dy = 0. \quad (162)$$

Reflectionless potentials

Suppose that $b(k) = 0$ and we have only two discrete eigenvalues $(N = 2)$:

$$\psi_1(x) \sim c_1 \exp\left(-\kappa_1 x\right), \quad \psi_2 \sim c_2 \exp\left(-\kappa_2 x\right) \quad \text{as } x \to \infty, \tag{2}$$

$\kappa_1 \neq \kappa_2$.

Then, we obtain

$$F(X) = c_1^2 \exp(-\kappa_1 X) + c_2^2 \exp(-\kappa_2 X). \tag{3}$$

The Marchenko equation then becomes

$$K(x,z) + c_1^2 \exp\left[-\kappa_1(x+z)\right] + c_2^2 \exp\left[-\kappa_2(x+z)\right]$$
$$+ \int_x^\infty K(x,y)\left\{c_1^2 \exp\left[-\kappa_1(y+z)\right] + c_2^2 \exp\left[-\kappa_2(y+z)\right]\right\} dy = 0 \tag{4}$$

$F(X)$ is obviously separable, therefore we set

$$K(x,z) = L_1(x)\exp\left(-\kappa_1 z\right) + L_2(x)\exp\left(-\kappa_2 z\right), \tag{6}$$

i.e. $X_n(x) = c_n^2 \exp(-\kappa_n x)$ and $Z_n(z) = \exp(-\kappa_n z)$

It follows that $L_1(x)$ and $L_2(x)$ must satisfy

$$
\begin{aligned}
&L_1 + c_1^2 \exp(-\kappa_1 x) \\
&+ c_1^2 \left\{ L_1 \int_x^\infty \exp\left(-2\kappa_1 y\right) dy + L_2 \int_x^\infty \exp\left[-(\kappa_1+\kappa_2)y\right] dy \right\} = 0 \\
&L_2 + c_2^2 \exp(-\kappa_2 x) \\
&+ c_2^2 \left\{ L_1 \int_x^\infty \exp\left[-(\kappa_1+\kappa_2)y\right] dy + L_2 \int_x^\infty \exp\left(-2\kappa_2 y\right) dy \right\} = 0.
\end{aligned}
\tag{8}
$$

After having evaluated the integrals the system becomes

$$L_n + c_n^2 \exp(-\kappa_n x) + c_n^2 \sum_{m=1}^{2} \frac{L_m \exp\left[-(\kappa_m + \kappa_n)x\right]}{\kappa_m + \kappa_n} = 0, \quad n = 1, 2. \tag{10}$$

The above system can be written as $AL + B = 0,$ where

$L = (L_1, L_2)$ and

$$B_n = c_n^2 \exp(-\kappa_n x) \tag{11}$$

The matrix A is a square matrix with elements

$$A_{mn} = \delta_{mn} + c_m^2 \frac{\exp\left[-(\kappa_m + \kappa_n)x\right]}{\kappa_m + \kappa_n}, \qquad (13)$$

where δ_{mn} is the Kronecker delta.

The solution for L is therefore

$$L = -A^{-1}B. \qquad (14)$$

Moreover, $K(x, x) = E^T L$ where $E_n = \exp(-\kappa_n x)$.

We now note that

$$\frac{d}{dx}A_{mn} = -c_m^2 \exp\left[-(\kappa_m + \kappa_n)x\right] = -B_m E_n. \tag{15}$$

Therefore, we obtain

$$\begin{aligned} K(x,x) &= \sum_{m=1}^{N} E_m L_m = -\sum_{m,n=1}^{N} E_m \left(A^{-1}\right)_{mn} B_n \\ &= \mathrm{Tr}\left(A^{-1}\frac{dA}{dx}\right). \end{aligned} \tag{17}$$

Now, for the 2×2 case if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{18}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{19}$$

After trivial algebra we obtain

$$A^{-1}\frac{dA}{dx} = \frac{1}{ad-bc}\begin{pmatrix} a'd - bc' & \cdots \\ \cdots & -cb' + ad' \end{pmatrix}. \qquad (20)$$

Finally, we have

$$\begin{aligned}\mathrm{Tr}\left(A^{-1}\frac{dA}{dx}\right) &= \frac{1}{ad-bc}(a'd + d'a - b'c - bc') \\ &= \frac{1}{\det A}\frac{d\det A}{dx} = \frac{d}{dx}\log\det A. \qquad (22)\end{aligned}$$

We can now evaluate $u(x)$

$$u(x) = -2\frac{d\hat{K}}{dx} = -2\frac{d^2}{dx^2}\log\det A. \quad (24)$$

In the case with just two discrete eigenvalues, we have

$$\det A = \left[1 + \frac{c_1^2}{2k_1}\exp(-2\kappa_1 x)\right]\left[1 + \frac{c_2^2}{2k_2}\exp(-2\kappa_2 x)\right] - \frac{c_1^2 c_2^2}{(\kappa_1+\kappa_2)^2}\exp\left[-2(\kappa_1+\kappa_2)x\right]. \quad (25)$$

If we set $c_2 = 0$, we then obtain

$$u(x) = -\frac{4\kappa_1 c_1^2 \exp(-2\kappa_1 x)}{\left[1 + \frac{c_1^2}{2\kappa_1}\exp(-2\kappa_1 x)\right]^2} \qquad (27)$$
$$= -2\kappa_1^2 \operatorname{sech}^2\left[\kappa_1 x + x_0\right],$$

where $\exp(x_0) = (2\kappa_1)^{1/2}/c_1$. If $\kappa_1 = 1$ and $c_1 = \sqrt{2}$, then we recover

$$u(x) = -2\operatorname{sech}^2 x. \qquad (28)$$

Reflection coefficient with one pole

Suppose the scattering data are given by

$$b(k) = -\frac{\beta}{\beta + ik} \quad \text{and} \quad \psi(x) \sim \beta^{1/2} e^{-\beta x} \quad \text{as } x \to \infty \qquad (30)$$

where $\beta > 0$. $b(k)$ has a simple pole at $k = i\beta$, therefore there is one discrete eigenvalue which is $\kappa_1 = \beta$ and $c_1 = \beta^{1/2}$. We then have

$$F(X) = \beta e^{-\beta X} - \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikX}}{\beta + ik} dk. \qquad (31)$$

The integral can be calculated easily using Cauchy's residue theorem:

$$\int_{-\infty}^{\infty} \frac{e^{ikX}}{\beta + ik} dk = 2\pi e^{-\beta X}, \quad X > 0 \tag{32}$$

and

$$\int_{-\infty}^{\infty} \frac{e^{ikX}}{\beta + ik} dk = 0 \quad X < 0. \tag{33}$$

$F(X)$ becomes simply

$$F(X) = \beta e^{-\beta X} H(-X), \quad (35)$$

where $H(-X)$ is the Heaviside step function.

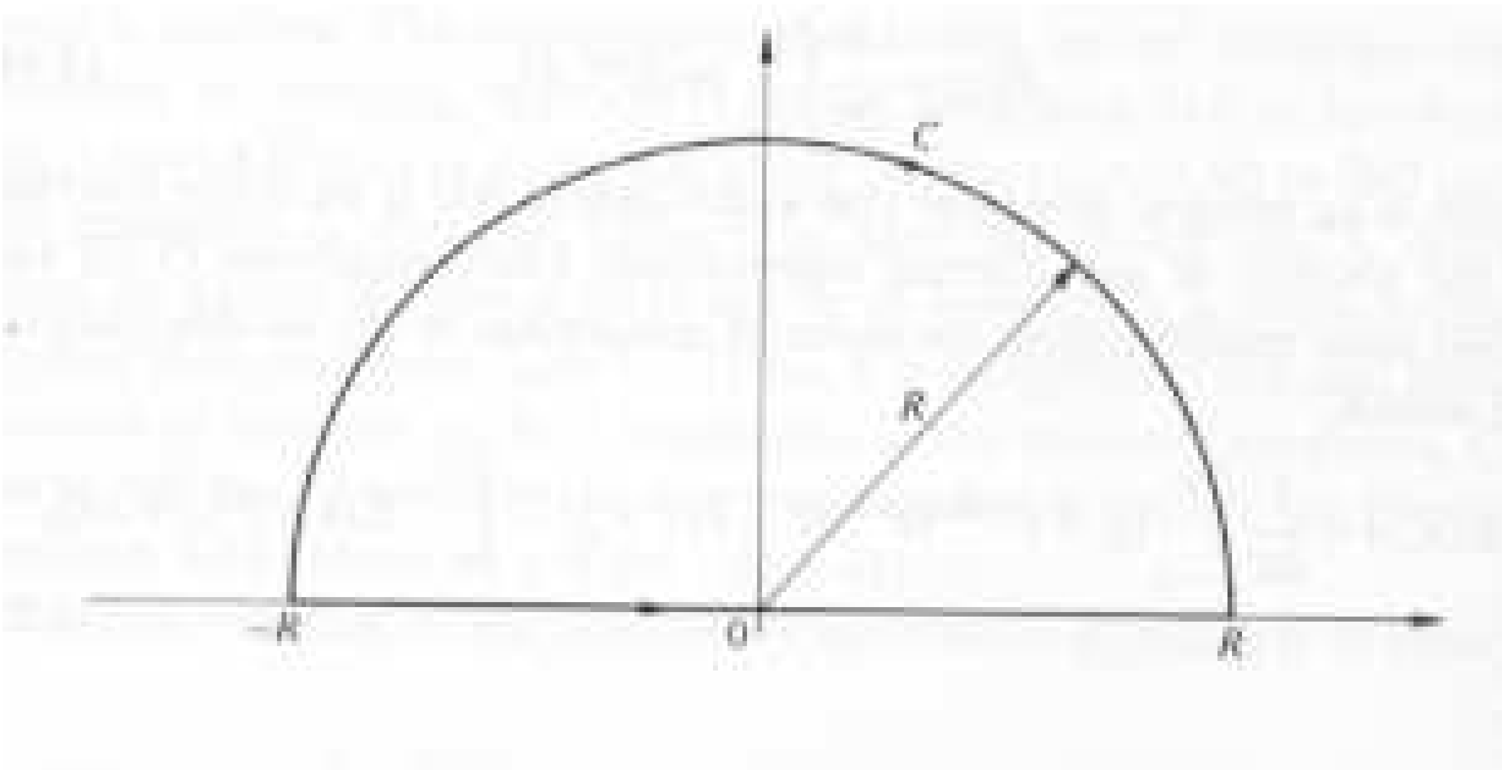


Figure 1: The contour of the integral (32) in the complex plain..

From the Marchenko equation

$$K(x,z)+F(x+z)+\int_x^{\infty} K(x,y)F(y+z)dy = 0 \quad z > x > -\infty, \tag{36}$$

it follows that

$$K(x,z) = 0 \quad \text{for } x+z > 0. \tag{37}$$

The Marchenko equation now becomes, for $x+z<0$,

$$K(x,z) + \beta e^{-\beta(x+z)} + \beta \int_x^{-z} K(x,y)e^{-\beta(y+z)}dy = 0 \tag{39}$$

Integrating by parts (on remembering that $F(y+z)=0$ for $y+z>0$) yields

$$K(x,z)+\beta e^{-\beta(x+z)}+K(x,x)e^{-\beta(x+z)}-K(x,-z) + \int_x^{-z} K_y(x,y)e^{-\beta(y+z)}dy = 0. \quad (40)$$

The solution is

$$\boxed{K(x,z) = -\beta.}$$

Finally, we have

$$K(x, z) = -\beta H(-x - z) \quad \text{and so} \quad K(x, x) = -\beta H(-2x).$$

The required potential is therefore

$$u(x) = 2\beta \frac{d}{dx} H(-2x) = -2\beta\delta(x). \tag{42}$$

It coincides with the previous example by setting $\beta = V/2$.

4. The initial-value problem for the KdV equation

Recapitulation

The potential function $u(x)$ for the Sturm-Liouville equation

$$\psi'' + (\lambda - u(x))\psi = 0, \quad -\infty < x < \infty \qquad (44)$$

can be reconstructed from the scattering data.

Scattering data:

- Discrete spectrum ($\lambda < 0$):

$$\psi_n(x), \quad \lambda_n = -\kappa_n^2, \quad n = 1, \ldots, N \tag{48}$$

$$\psi_n(x) \sim c_n \exp(-\kappa_n x) \quad \text{as } x \to \infty. \tag{49}$$

- Continuum spectrum ($\lambda > 0$):

$$\psi(x;k) \sim \begin{cases} e^{-ikx} + b(k)e^{ikx} & \text{as } x \to \infty \\ a(k)e^{-ikx} & \text{as } x \to -\infty. \end{cases} \tag{50}$$

$k = \lambda^{1/2}$, $a(k)$ and $b(k)$ are the transmission and reflection coefficients respectively: $|b(k)|^2 + |a(k)|^2 = 1$.

Then we showed that

$$u(x) = -2\frac{dK(x,x)}{dx}, \quad (52)$$

where $K(x,z)$ is the solution to the ***Marchenko equation***

$$K(x,z)+F(x+z)+\int_x^\infty K(x,y)F(y+z)dy = 0 \quad z > x > -\infty, \quad (53)$$

$K(x,z) = 0$ if $z < x$. Moreover, $F(X)$ is defined by

$$F(X) = \sum_{n=1}^{N} c_n^2 \exp\left(-\kappa_n X\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k)e^{ikX}dk. \quad (54)$$

Inverse scattering and the KdV equation

Consider the

$$u_t - 6uu_x + u_{xxx} = 0.$$

Introducing the ***Miura transformation***

$$u = v^2 + v_x,$$

the KdV equation becomes

$$\left(2v + \frac{\partial}{\partial x}\right)(v_t - 6v^2 v_x + v_{xxx}) = 0. \tag{55}$$

By making the substitution $v = \psi_x/\psi$ and by applying the transformation

$$u \to \lambda + u(x + 6\lambda t, t), \quad -\infty < \lambda < \infty. \tag{56}$$

the Miura transformation becomes

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0, \quad -\infty < x < \infty. \tag{57}$$

$\psi(x;t)$, and therefore also $\lambda(t)$, depend parametrically on time because $v(x,t)$ is a solution of the mKdV equation (55) and therefore $u(x,t)$ of the KdV equation.

Inverse scattering transform for the KdV equation

1. Initial-value problem $u(x,0) = f(x)$;
2. solve the scattering problem with potential $f(x)$, and determine the scattering data $S(0)$ ($c_n(0)$, $\kappa_n(0)$ and $b(k;0)$) at $t = 0$;
3. determine the time evolution of the scattering data $S(t)$;
4. reconstruct $u(x,t)$ solving the inverse scattering problem for $S(t)$.

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{scattering}} & S(0) \\
\text{KdV}\Big\downarrow & & \Big\downarrow\text{time evolution} \\
u(x,t) & \xleftarrow[\text{scattering}]{\text{inverse}} & S(t)
\end{array}
$$

The diagram of the inverse scattering for the KdV equation.

Time evolution of the scattering data

We begin once again from the Sturm-Liouville problem for $\psi(x;t)$:

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0, \quad -\infty < x < \infty. \tag{58}$$

We then differentiate the above equation w.r.t. x,

$$\psi_{xxx} - u_x\psi + (\lambda - u)\psi_x = 0, \tag{59}$$

and w.r.t. t,

$$\psi_{xxt} + (\lambda_t - u_t)\psi + (\lambda - u)\psi_t = 0. \tag{60}$$

Here $u(x,t)$ satisfies the KdV equation.

It is now convenient to define

$$R(x,t) = \psi_t + u_x\psi - 2(u + 2\lambda)\psi_x. \qquad (62)$$

We now construct the identity

$$\frac{\partial}{\partial x}\left(\psi_x R - \psi R_x\right) = \psi_{xx}\left(\psi_t + u_x\psi - 2u\psi_x - 4\lambda\psi_x\right)$$
$$- \psi(\psi_{xxt} + u_{xxx}\psi - 3u_x\psi_{xx} - 2u\psi_{xxx} - 4\lambda\psi_{xxx}). \quad (63)$$

The next step consists of eliminating ψ_{xxt} and ψ_{xxx} using equations (59) and (60).

This leads to

$$\frac{\partial}{\partial x}\left(\psi_x R - \psi R_x\right) = \psi_{xx}\left(\psi_t - 2u\psi_x - 4\lambda\psi_x\right) - \psi\left(u_{xxx}\psi - 4u_x\psi_{xx}\right)$$
$$-\psi\left(u\psi_t - \lambda\psi_t - \lambda_t\psi + u_t\psi\right) + \psi\left(2u + 4\lambda\right)\left(u_x\psi - \lambda\psi_x + u\psi_x\right). \tag{64}$$

We now use the Schrödinger equation

$$\psi_{xx} + (\lambda - u(x,t))\psi = 0 \tag{65}$$

to simplify (64).

This yields

$$\frac{\partial}{\partial x}\left(\psi_x R - \psi R_x\right) = \psi^2\left(\lambda_t - u_t + 6uu_x - u_{xxx}\right). \qquad (66)$$

Finally, since $u(x,t)$ satisfies the KdV equation, we have

$$\frac{\partial}{\partial x}\left(\psi_x R - \psi R_x\right) = \lambda_t \psi^2. \qquad (68)$$

This equation can now be used to obtain the time evolution of the scattering data.

The discrete spectrum

We now choose $\lambda = -\kappa_n^2$ and $\psi = \psi_n$, $n = 1, \ldots N$. By integrating the equation

$$\frac{\partial}{\partial x}\left(\psi_{nx} R_n - \psi_n R_{nx}\right) = -(\kappa_n)_t^2 \psi_n^2 \tag{69}$$

w.r.t. x, we have

$$\left[\psi_{nx} R_n - \psi_n R_{nx}\right]_{-\infty}^{\infty} = -(\kappa_n)_t^2 \int_{-\infty}^{\infty} \psi_n^2(x) dx = -(\kappa_n)_t^2. \tag{70}$$

Now the L.H.S. is zero, therefore

$$(\kappa_n)_t^2 = 0 \quad \text{or} \quad \kappa_n = \text{constant}.$$

We now want to determine the time evaluation of c_n.

Integrating (69) w.r.t. x gives

$$\psi_{nx} R_n - \psi_n R_{nx} = g_n(t), \tag{71}$$

where $g_n(t)$ are arbitrary functions of t. But

$$R_n, \psi_n \to 0 \quad \text{as } |x| \to \infty. \tag{72}$$

$$\text{Therefore } \boxed{\; g_n(t) = 0 \;} \; \forall n, t.$$

We now integrate again

$$\psi_{nx} R_n - \psi_n R_{nx} \tag{73}$$

and obtain (using integration by parts)

$$R_n / \psi_n = h_n(t), \tag{74}$$

where $h_n(t)$ $(n = 1, 2, \ldots, N)$ are arbitrary functions too.

By multiplying (74) by ψ_n^2, we obtain

$$\psi_n(\psi_{nt} + u_x \psi_n - 2u\psi_{nx} + 4\kappa_n^2 \psi_{nx}) = h_n \psi_n^2 \tag{75}$$

or, using $\psi_{nxx} - (\kappa_n^2 + u(x,t))\psi_n = 0$

$$\frac{1}{2} (\psi_n)_t^2 + \left(u\psi_n^2 - 2\psi_{nx}^2 + 4\kappa_n^2 \psi_n^2\right)_x = h_n \psi_n^2. \tag{76}$$

We now integrate (76) w.r.t. x:

$$\frac{1}{2}\frac{d}{dt}\left(\int_{-\infty}^{\infty}\psi_n^2 dx\right) = h_n \int_{-\infty}^{\infty}\psi_n^2 dx \tag{77}$$

This implies that $\boxed{h_n(t) = 0}$ $\forall n, t$.

It follows that

$$\boxed{R_n = \psi_{nt} + u_x\psi_n - 2\left(u - 2\kappa_n^2\right)\psi_{nx} = 0.}$$

This is the time-evolution equation for $\psi_n(x;t)$.

The previous equation (red box) can be used to find the time evolution of the $c_n(t)$.

We know that

$$u \to 0 \quad \text{and} \quad \psi_n(x;t) \sim c_n(t)\exp(-\kappa_n x) \quad \text{as } x \to \infty. \tag{78}$$

This asymptotic behaviour inserted in the equation for $\psi_n(x;t)$ (red box) gives

$$\frac{dc_n}{dt} - 4\kappa_n^3 c_n = 0 \quad \text{or} \quad c_n(t) = c_n(0)\exp(4\kappa_n^3 t), \tag{80}$$

where $c_n(0)$ are the normalization constants determined at $t = 0$.

The continuous spectrum

We start again from equation

$$\frac{\partial}{\partial x}\left(\psi_x R - \psi R_x\right) = \lambda_t \psi^2. \tag{82}$$

By integrating w.r.t. x over $\mathbb{R}$ we obtain that

$$\lambda_t = 0 \quad \text{or} \quad \lambda = \text{constant}.$$

By integrating once Eq. (82) gives

$$\hat{\psi}_x \hat{R} - \hat{\psi} \hat{R}_x = g(t;k),$$

where $\hat{R}$ is R evaluated in terms of $\hat{\psi}$ and $g(t;k)$ is a function of integration.

R is defined by

$$R(x,t) = \psi_t + u_x \psi - 2(u + 2\lambda)\psi_x. \tag{83}$$

For the continuous eigenfunctions we have

$$\hat{\psi}(x;t,k) \sim a(k;t)e^{-ikx} \quad \text{as } x \to -\infty \tag{84}$$

Therefore, we have that

$$\hat{R}(x,t;k) \sim \left(\frac{da}{dt} + 4ik^3 a\right) e^{-ikx}, \quad x \to -\infty. \quad (86)$$

As a consequence

$$\hat{\psi}_x \hat{R} - \hat{\psi}\hat{R}_x \to 0, \quad \text{as } x \to -\infty. \tag{87}$$

Thus $g(t;k) = 0$ for all t.

By integrating (87) once more we have

$$\hat{R}/\hat{\psi} = h(t;k) \quad \text{or} \quad \hat{R} = h\hat{\psi}. \tag{88}$$

Now, using again

$$\hat{\psi}(x;t,k) \sim a(k;t)e^{-ikx} \quad \text{as } x \to -\infty \tag{89}$$

we have that

$$\frac{da}{dt} + 4ik^3 a = ha. \tag{91}$$

The behaviour of $\hat{R}$ as $x \to \infty$ is given by

$$\hat{R}(x,t;k) \sim \frac{db}{dt}e^{ikx} + 4ik^3(e^{-ikx} - e^{ikx}) \quad \text{as } x \to \infty. \tag{92}$$

where we have used

$$\hat{\psi} \sim e^{-ikx} + b(k;t)e^{ikx}. \tag{93}$$

Substituting (92) into $\hat{R} = h\hat{\psi}$ yields

$$\frac{db}{dt}e^{ikx} + 4ik^3(e^{-ikx} - e^{ikx}) = h(e^{-ikx} - e^{ikx}). \tag{94}$$

Because e^{ikx} and e^{-ikx} are linearly independent the above equation implies

$$\frac{db}{dt} - 4ik^3 b = hb \quad \text{and} \quad h(t;k) = 4ik^3. \tag{95}$$

Finally, we have

$$\frac{da}{dt} = 0 \quad \text{and} \quad \frac{db}{dt} = 8ik^3 b \tag{96}$$

whose solutions are

$$a(k;t) = a(k;0) \qquad \text{and} \qquad b(k;t) = b(k;0)\exp(8ik^3 t).$$

Evolution of the scattering data: summary

- $\kappa_n = \text{constant}, \; c_n(t) = c_n(0) \exp(4\kappa_n^3 t);$
- $b(k;t) = b(k;0) \exp(8ik^3 t).$

Construction of the solution of the KdV equation: summary

We want to integrate the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \quad t > 0, \quad -\infty < x < \infty,$$

with initial condition $$u(x,0) = f(x).$$

Step 1: We solve the Sturm-Liouville equation

$$\psi_{xx} + (\lambda - u)\,\psi = 0, \quad -\infty < x < \infty.$$

with $$u(x,0) = f(x).$$

i.e. We determine the discrete spectrum $-\kappa_n^2$*, the normalization constants* $c_n(0)$*, and the reflection coefficient* $b(k;0)$*.*

Step 2: The time evolution of the scattering data is given by

- $\kappa_n = \text{constant};$
- $c_n(t) = c_n(0)\exp(4\kappa_n^3 t);$
- $b(k;t) = b(k;0)\exp(8ik^3 t).$

Step 3: We now want to solve the Marchenko equation

$$K(x,z;t) + F(x+z;t) + \int_x^\infty K(x,y;t)F(y+z;t)dy = 0, \quad (98)$$

with

$$F(X;t) = \sum_{n=1}^{N} c_n(0)^2 \exp\left(8\kappa_n^3 t - \kappa_n X\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0)\exp(8ik^3t + ikX)dk \quad (99)$$

Finally, the solution of the KdV equation can be expressed as

$$u(x,t) = -\frac{\partial}{\partial x}\hat{K}(x,t) \quad \text{and} \quad \hat{K}(x,t) = K(x,x,t). \qquad (101)$$

We have reduced the solution of a ***nonlinear partial*** *differential equation to that of solving* ***two linear*** *problems (a second order ODE and an integral equation).*

4.1 Reflectionless potentials

Solitary wave

We obtain the solitary wave by posing a suitable initial-value problem, without the assumption that the solution takes the form of a steady progressing wave.

The initial profile is taken to be

$$u(x,0) = -2\,\operatorname{sech}^2 x.$$

The Sturm-Liouville equation at $t = 0$ is

$$\psi_{xx} + \left(\lambda + 2\,\mathrm{sech}^2\, x\right)\psi = 0.$$

We have already studied this scattering problem. We have

$b(k) = 0$ and the discrete spectrum has only one eigenvalue

$$\kappa_1 = 1 \quad \text{and} \quad \psi_1(x) = 2^{-1/2}\,\mathrm{sech}\, x.$$

Moreover, we have

$$\psi(x) \sim 2^{1/2} e^{-x}, \quad x \to \infty, \quad \text{so} \quad \boxed{c_1(0) = 2^{1/2}}. \tag{102}$$

Now we have $c_1(t) = 2^{1/2}e^{4t}$, and therefore

$$F(X;t) = 2e^{8t-X}.$$

The Marchenko equation now becomes

$$K(x,z;t) + 2e^{8t-(x+z)} + 2\int_x^{\infty} K(x,y;t)e^{8t-(y+z)}dy = 0. \quad (104)$$

Since $F(X;t)$ is separable we can set $K(x,z;t) = L(x,t)e^{-z}.$

Therefore, the equation for $L(x,t)$ becomes

$$L + 2e^{8t-x} + 2Le^{8t}\int_x^\infty e^{-2y}dy = 0. \tag{105}$$

The above equation can be solved directly to yield

$$L(x,t) = -\frac{-2e^{8t-x}}{1+e^{8t-2x}}. \tag{107}$$

The potential is then given by

$$
\begin{aligned}
u(x,t) &= 2\frac{\partial}{\partial x}\left(\frac{2e^{8t-2x}}{1+e^{8t-2x}}\right) \\
&= -\frac{8e^{2x-8t}}{\left(1+e^{2x-8t}\right)^2} \\
&= -2\,\mathrm{sech}^2(x-4t).
\end{aligned}
\tag{109}
$$

This is the solitary wave solution of amplitude -2 *at speed of propagation* 4.

Two solitons solution

The initial profile is now taken to be

$$u(x,0) = -6\,\mathrm{sech}^2\,x.$$

The Sturm-Liouville equation is now

$$\psi_{xx} + \left(\lambda + 6\,\mathrm{sech}^2\,x\right)\psi = 0.$$

The discrete spectrum of this equation is given by

$$\kappa_1 = 1 \quad \text{and} \quad \kappa_2 = 2 \tag{112}$$

$$\psi_1(x) = \sqrt{\left(\frac{3}{2}\right)} \tanh x \operatorname{sech} x \quad \text{and} \quad \psi_2(x) = \frac{\sqrt{3}}{2} \operatorname{sech}^2 x \tag{113}$$

The asymptotic behaviour of these solutions is given by

$$\psi(x) \sim \sqrt{6} e^{-x}, \quad 2\sqrt{3} e^{-2x} \quad \text{as } x \to \infty. \tag{114}$$

Therefore, we have

$$c_1(0) = \sqrt{6} \qquad c_2(0) = 2\sqrt{3} \tag{115}$$

$$c_1(t) = \sqrt{6}e^{4t} \qquad c_2(t) = 2\sqrt{3}e^{32t} \tag{116}$$

This potential is reflectionless, therefore $b(k;0) = b(k;t) = 0.$

The function $F(X;t)$ is now

$$F(X;t) = 6e^{8t-X} + 12e^{64t-2X}. \tag{118}$$

The Marchenko equation is therefore

$$K(x,z;t)+6e^{8t-(x+z)}+12e^{64t-2(x+z)} + \int_x^\infty K(x,y;t)\left(6e^{8t-(y+z)}+12e^{64t-2(y+z)}\right)dy=0 \quad (119)$$

The solution $K(x,z;t)$ must take the form

$$K(x,z;t)=L_1(x,t)e^{-z}+L_2(x,t)e^{-2z} \quad (120)$$

Inserting the above expression into (119) and collecting the coefficients of e^{-z} and e^{-2z}, we obtain the pair of equations

$$L_1 + 6e^{8t-x} + 6e^{8t}\left(L_1\int_x^{\infty} e^{-2y}dy + L_2\int_x^{\infty} e^{-3y}dy\right) = 0$$
$$L_2 + 12e^{64t-x} + 12e^{64t}\left(L_1\int_x^{\infty} e^{-3y}dy + L_2\int_x^{\infty} e^{-4y}dy\right) = 0 \tag{122}$$

Evaluating the integrals yields

$$L_1 + 6e^{8t-x} + 3L_1e^{8t-2x} + 2L_2e^{8t-3x} = 0 \tag{123}$$
$$L_2 + 12e^{64t-2x} + 4L_1e^{64t-3x} + 3L_2e^{64t-4x} = 0. \tag{124}$$

The previous system can be easily solved to give

$$L_1(x,t) = 6\left(e^{72t-5x} - e^{8t-x}\right)/D \tag{125}$$

$$L_2(x,t) = -12\left(e^{64t-2x} + e^{72t-4x}\right)/D, \tag{126}$$

where $D = 1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}$.

The solution $u(x,t)$ to the KdV equation then becomes

$$\begin{aligned} u(x,t) &= -2\frac{\partial}{\partial x}\left(L_1 e^{-x} + L_2 e^{-2x}\right) \\ &= 12\frac{\partial}{\partial x}\left[\left(e^{8t-2x} + e^{72t-6x} - 2e^{64t-4x}\right)\right]/D \end{aligned} \tag{127}$$

After a little bit of manipulation it can be simplified to give

$$u(x,t) = -12\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}. \qquad (129)$$

We now want to look at the behaviour of this solution as $t \to \pm\infty$.

We now set $\xi = x - 16t$, *i.e.* we follow a wave which moves at speed 16 (if it exists):

$$u(x,t) = -12\frac{3 + 4\cosh(2\xi + 24t) + \cosh(4\xi)}{\left[3\cosh(\xi - 12t) + \cosh(3\xi + 12t)\right]^2}. \tag{130}$$

Taking the limit as $t \to \pm\infty$ yields

$$u(x,t) \sim -8\,\mathrm{sech}^2\left(2\xi \mp \tfrac{1}{2}\log 3\right) \quad \text{as } t \to \pm\infty, \quad \xi = x - 16t.$$

Similarly, by setting $\eta = x - 4t$

$$u(x,t) \sim -2\,\mathrm{sech}^2\left(2\eta \pm \tfrac{1}{2}\log 3\right) \quad \text{as } t \to \pm\infty, \quad \xi = x - 4t.$$

The last two expression can be combined (the error terms are asymptotically small) to give

$$u(x,t) \sim -8\,\mathrm{sech}^2\left(2\xi \mp \frac{1}{2}\log 3\right) \\ \qquad - 2\,\mathrm{sech}^2\left(2\eta \pm \frac{1}{2}\log 3\right) \quad \text{as } t \to \pm\infty.$$

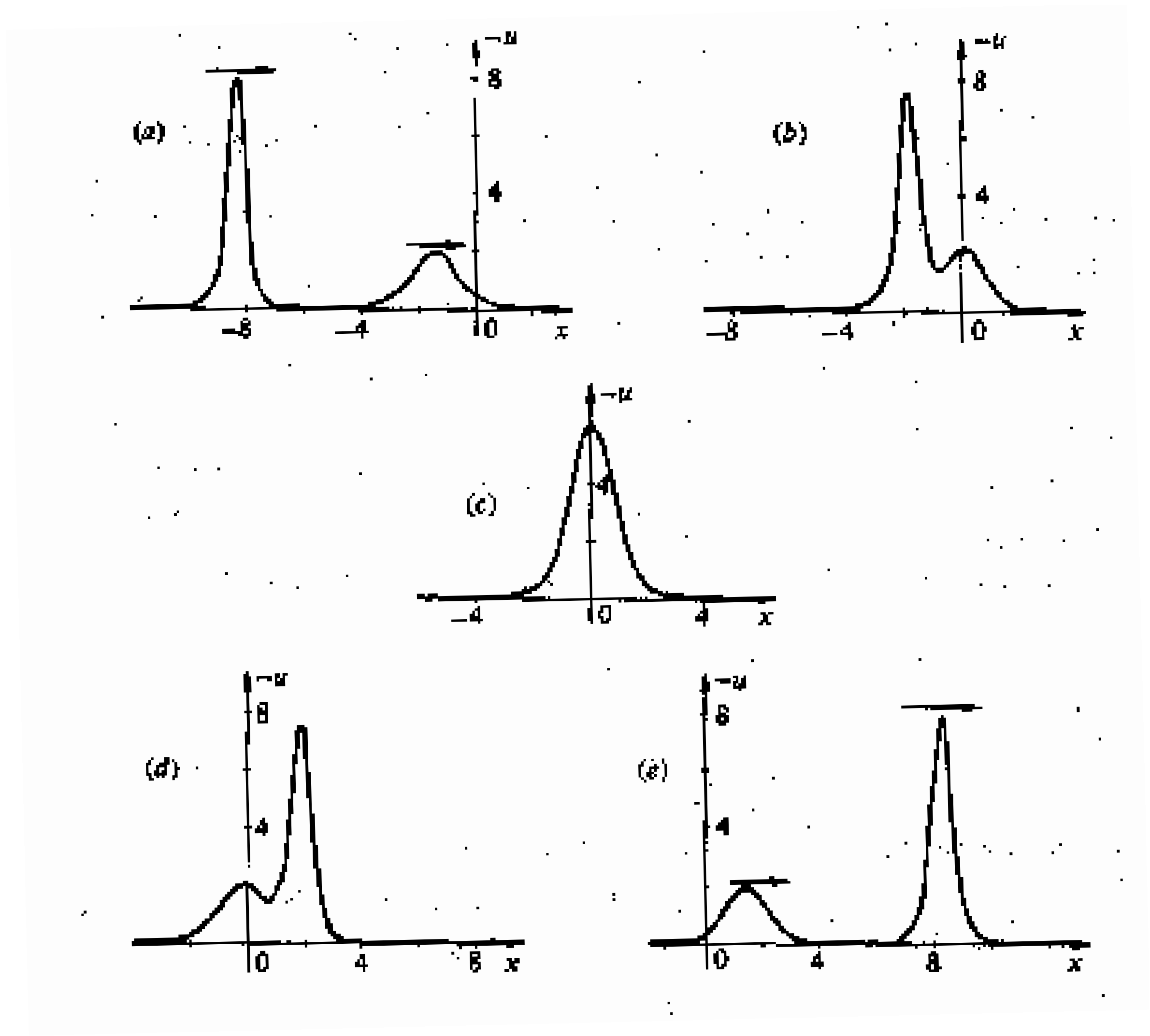


Figure 2: Time evolution of the two solitons solution

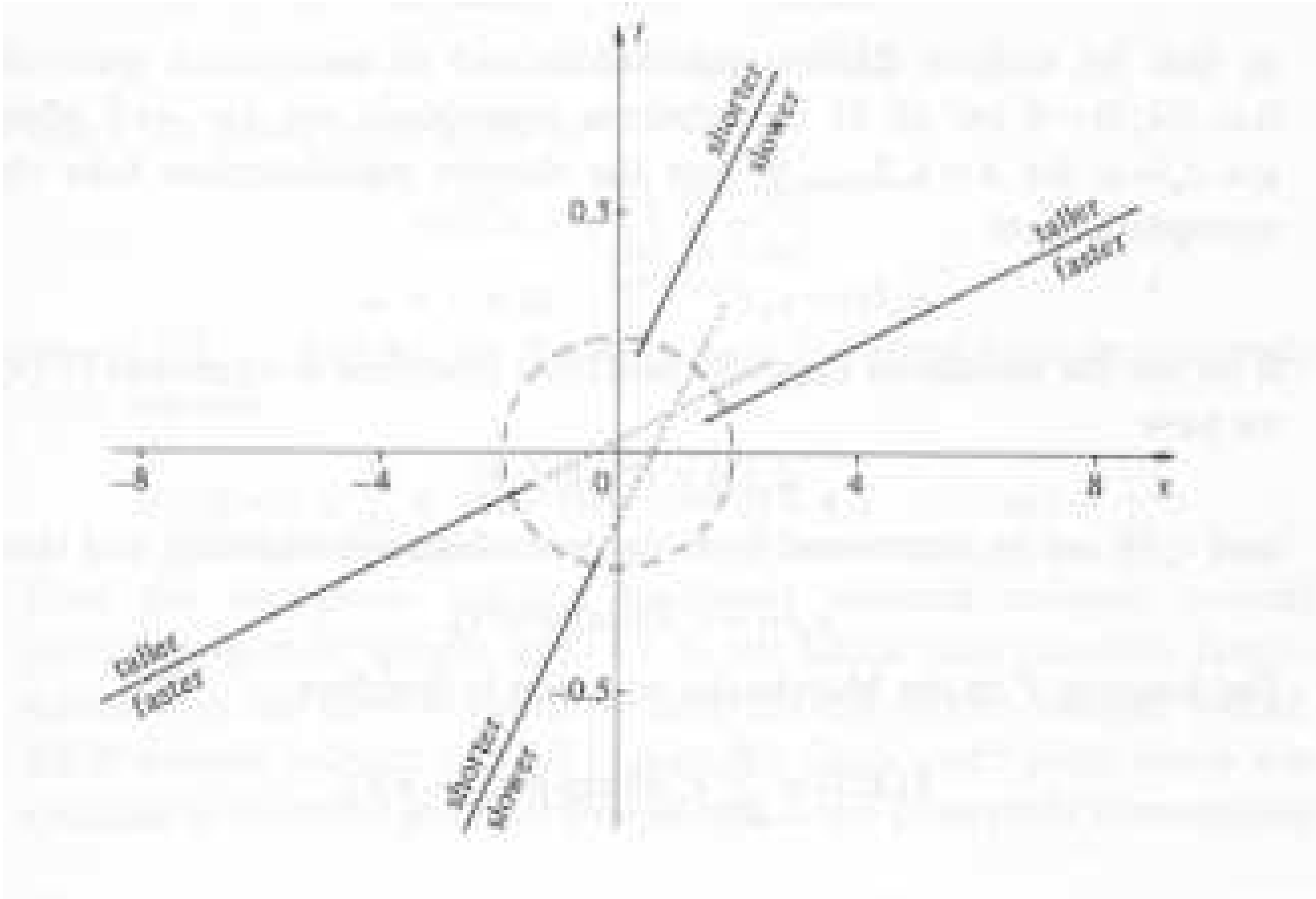


Figure 3: The paths of the wave crests of the two solitons

Comments

- As $t \to -\infty$ the solution ha the form of two solitons, the taller travelling faster;
- the taller catches the smaller, they coalesces, they form our initial profile at $t = 0$ and then the taller moves away;
- as $t \to \infty$ we have two solitons again;
- trace of the nonlinear interaction: after the interaction the two waves are phase shifted. The taller wave has moved ***forward***, the shorter ***backwards***.

N solitons solution

The initial profile is given by

$$u(x,0) = -N(N+1)\,\mathrm{sech}^2\,x.$$

The Sturm-Liouville equation is now

$$\psi_{xx} + \left[\lambda + N(N+1)\,\mathrm{sech}^2\,x\right]\psi = 0.$$

The discrete spectrum of this equation is given by

$$\kappa_n = n, \quad n = 1, \ldots N \tag{3}$$

$$\psi_n(x) \propto P_N^n(\tanh x) \tag{4}$$

The discrete eigenfunction take the asymptotic form

$$\psi_n \sim c_n e^{-nx} \quad \text{as } x \to \infty, \tag{5}$$

where c_n can be found using the normalization condition.

The time evolution of the normalization coefficients is given by

$$c_n(t) = c_n(0) \exp\left(4n^3 t\right) \tag{6}$$

The function $F(X;t)$ is now

$$F(X;t) = \sum_{n=1}^{N} c_n(0)^2 \exp\left(8n^3 t - nX\right) . \tag{8}$$

The Marchenko equation now becomes

$$K(x,z;t) + \sum_{n=0}^{N} c_n(0)^2 \exp\left[8n^3 - n(x+z)\right]$$
$$+ \int_x^\infty K(x,y;t) \sum_{n=1}^{N} c_n^2(0) \exp\left[8n^3 t - n(y+z)\right] dy = 0. \quad (9)$$

The solution for $K(x,z;t)$ must now take the form

$$K(x,z;t) = \sum_{n=1}^{N} L_n(x,t) e^{-nx}. \quad (10)$$

Inserting $K(x, z; t)$ into (9) and collecting the coefficients of e^{nz} the integral equation is replaced by an algebraic system:

$$AL + B = 0.$$

L and B are the column vectors with elements L_n and

$$B_n = c_n(0)^2 \exp\left(8n^3 t - nx\right) \tag{11}$$

The $N \times N$ matrix A has elements

$$A_{mn} = \delta_{mn} + \frac{c_m^2(0)}{m+n} \exp\left[8m^3 t - (m+n)x\right]. \qquad (13)$$

We now from the inverse scattering theory of reflectionless potentials that

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \log \det A. \qquad (15)$$

The asymptotic form of the solution can be determined by setting

$$\xi_n = x - 4\kappa_n^2 t = x - 4n^2 t \tag{16}$$

and then taking the limit $t \to \pm\infty$. For fixed ξ_n we have

$$u(x,t) \sim -2n^2 \operatorname{sech}^2 \left[n \left(x - 4n^2 t \right) \mp x_n \right], \quad t \to \pm\infty. \tag{18}$$

The phase x_n is given by

$$\exp(2x_n) = \prod_{\substack{m=1 \\ m \neq n}} \left| \frac{n-m}{m+m} \right|^{\operatorname{sgn}(n-m)} \qquad n = 1, 2, \ldots, N \qquad (19)$$

We can combine the previous asymptotic solutions to obtain

$$u(x,t) \sim -2 \sum_{n=1}^{N} n^2 \operatorname{sech}^2 \left[n \left(x - 4n^2 t \right) \mp x_n \right], \quad t \to \pm\infty.$$

We can combine these solutions as if the equation were linear because the error that we make in doing so is exponentially small as $t \to \pm\infty$.

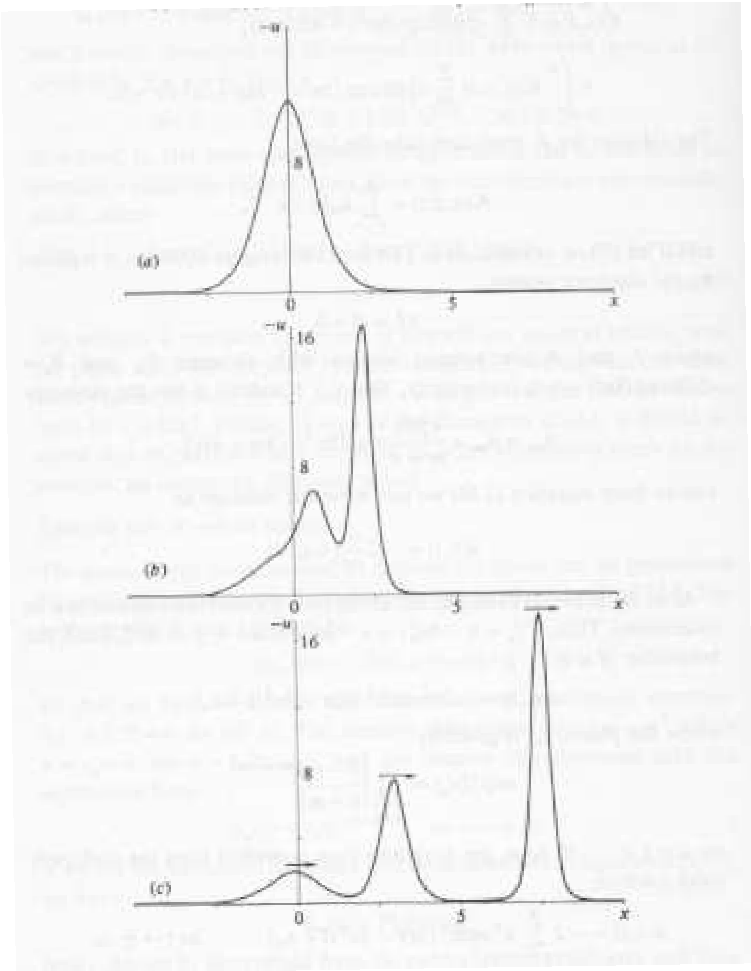


Figure 1: The three solitons solution with $u(x, 0) = -12\,\mathrm{sech}^2\, x$. (a) $t = 0$; (b) $t = 0.05$; (c) $t = 0.2$. $-u$ is plotted against x

4.2 General description of the solution

When $b(k) \neq 0$ the Marchenko equation cannot be solved for $K(x, z; t)$ in closed form.

In general the function $F(X; t)$ is

$$F(X;t) = \sum_{n=1}^{N} c_n(0)^2 \exp\left(8\kappa_n^3 t - \kappa_n X\right) + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0)\exp(8ik^3 t + ikX)dk. \quad (20)$$

We now consider the contribution to the solution $K(x, z; t)$ of the integral in the above expression.

The integral

$$I(X;t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k;0)\exp(8ik^3t + ikX)dk. \qquad (21)$$

is of the type

$$I(t) = \int_{-\infty}^{\infty} f(k)e^{i\phi(k)t}dk, \quad \phi(k) \in \mathbb{R} \qquad (23)$$

The leading order behaviour of this integral as $t \to \infty$ can be determined using the ***stationary phase approximation***.

The main contribution as $t \to \infty$ comes from where $\phi(k)$ is stationary and

$$I(t) \sim \frac{2\Gamma\left(\frac{1}{n}\right)(n!)^{1/n}}{n\left[t\left|\phi^{(n)}(c)\right|\right]^{1/n}} f(c)e^{it\phi(c)\pm i\pi/(2n)}, \quad t \to \infty. \qquad (25)$$

Here $\phi^{(n)}(c)$ is the first non zero derivative at the stationary point c, and we use the factor $e^{i\pi/(2n)}$ if $\phi^{(n)}(c) > 0$ and the factor $e^{-i\pi/(2n)}$ if $\phi^{(n)}(c) < 0$.

Therefore, our integral becomes

$$I(X;t) \sim \frac{\Gamma\left(1/3\right) e^{i\pi/6}}{\pi}\sqrt{\frac{2}{3}}b(0;0)t^{-1/3}, \quad t \to \infty \tag{26}$$

Let consider for a moment an initial condition $u(x,0)$ with no discrete spectrum (*e.g.* positive δ-function or positive sech^2 profile.)

$F(x,z;t)$ and therefore $K(x,z;t)$ are of order $O(t^{-1/3})$.

As a consequence *$u(x,t)$ is of order $O(t^{-1/3})$ too.*

Therefore, as $t \to \infty$, the term $-6uu_x$ in

$$u_t - 6uu_x + u_{xxx} = 0 \tag{27}$$

is negligible and therefore, ***locally***, we have

$$u_t \sim -u_{xxx}, \quad t \to \infty.$$

This is a linear equation whose dispersion relation is $\omega = -k^3$.

In the absence of the discrete spectrum $u(x,t)$ behaves like a wave packet propagating (almost) linearly to the left with group velocity $d\omega/dt = -3k^2$ and whose amplitude decays as $t^{-1/3}$.

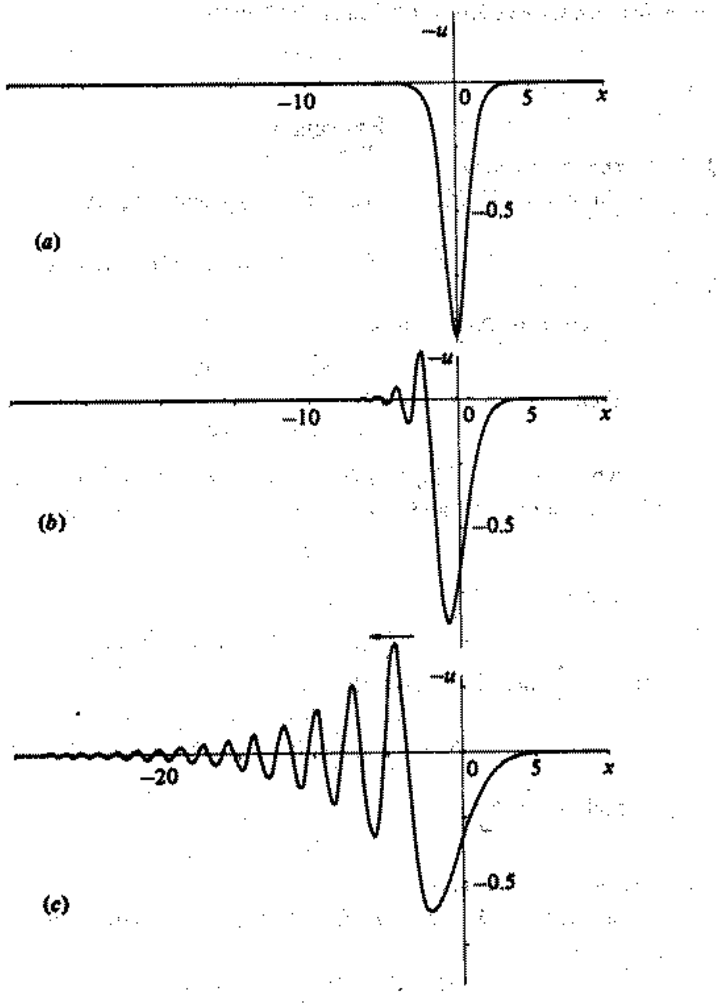


Figure 2: Solution to the Kdv equation with positive $\operatorname{sech}^2 x$ initial condition.

Now, let us consider a system whose initial condition has one discrete eigenvalue and $b(k) \neq 0$ (for example a negative δ-function).

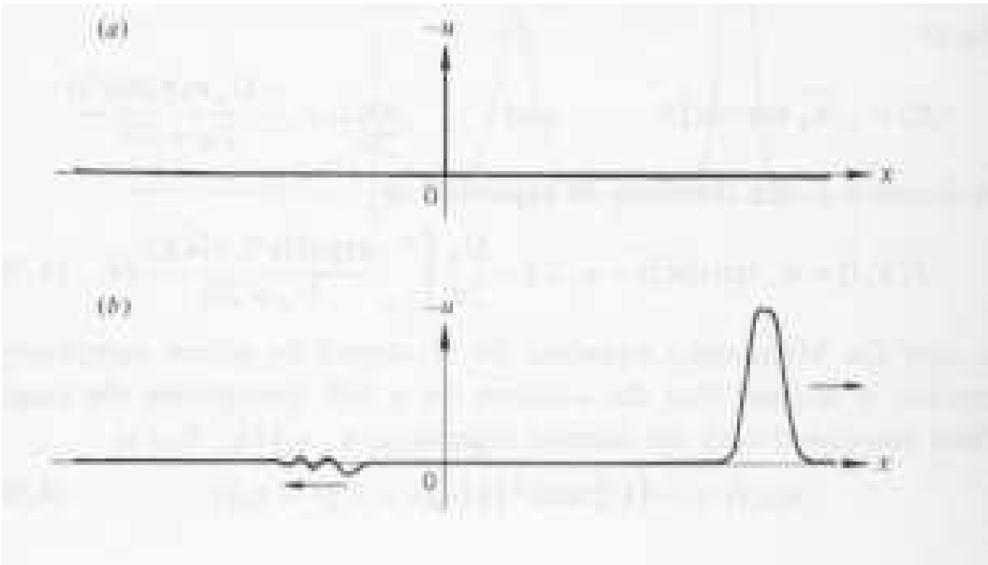


Figure 3: Initial condition with a δ profile.

Example: negative sech^2

Let us choose as initial condition

$$u(x,0) = -4\,\text{sech}^4\,x.$$

4 is *not* of the form $N(N+1)$, and therefore the reflection coefficient $b(k)$ is not zero.

The number of eigenvalues is given by

$$\left[\left(V+\frac{1}{4}\right)^{1/2}-\frac{1}{2}\right]+1. \tag{28}$$

Therefore, we have two discrete eigenvalues.

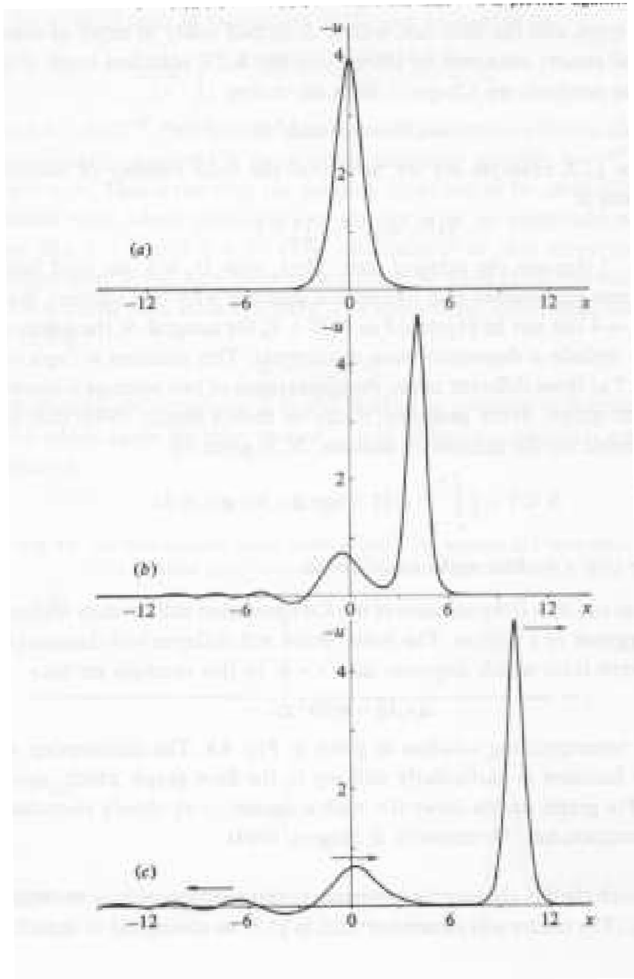


Figure 4: Initial condition given by $u(x,0) = -4\,\mathrm{sech}^4\,x$.

Recapitulation

- The solution $u(x,t)$ evolves for $t > 0$ so that $\kappa_n = \text{const.}$, $c_n(t) = c_n(0)\exp(4\kappa_n^3 t)$ and $b(k;t) = b(k;0)\exp(8ik^3 t)$.
- The solution separates into two parts as $t \to \infty$:
- (i) There is a precession of N solitary waves of depression. Each wave has positive velocity which is proportional to its amplitude. They are essentially nonlinear waves which interact like solitons, changing only their phases. They are associated with the discrete spectrum and with the sum $\sum_{n=1}^{N} c_n(t)\exp(-\kappa_n X)$ of $F(X;t)$ in the Marchenko equation. In the exceptional case of $b(k,0) = 0\ \forall k$, the precession of solitary waves is the complete solution as $t \to \infty$.

- (ii) If $b(k,0) \neq 0$ for some k, then there exists also an oscillatory wave train as $t \to \infty$. This train has $u(x,t)$ of both signs for each t. It is essentially linear, *i.e.* $u_t \sim -u_{xxx}$ locally as $t \to \infty$. It propagates to the left with group velocity $c_g = -3k^2 < 0$. The train is associated with the continuous spectrum and the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,t)e^{ikX}\,dk \tag{30}$$

of $F(X;t)$ in the Marchenko equation.

New definition of soliton

Def.: *A soliton is that component of the solution of a nonlinear evolution equation which depends* ***only upon one constant*** *discrete eigenvalue of the underlying scattering problem as* $t \longrightarrow \pm\infty$

This definition clarifies what is meant by the 'identity' of the soliton: it is that property which maintains the constancy of the discrete eigenvalues.

5. Conservation laws

Consider the ***equation of continuity*** of a compressible fluid,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \quad (32)$$

where $\rho(x,t)$ is the density and $u(x,t)$ is the x-velocity of the fluid.

The above equation expresses the ***conservation of mass of fluid***.

Now, suppose that $\rho u \to$ const. as $|x| \to \infty$, and ρ and $(\rho u)_x$ are integrable.

The continuity equation implies that

$$\frac{d}{dt}\int_{-\infty}^{\infty} \rho dx = \int_{-\infty}^{\infty} \frac{\partial \rho}{\partial t} dx = -\int_{-\infty}^{\infty} \frac{\partial (\rho u)}{\partial x} dx = -[\rho u]_{-\infty}^{\infty} = 0. \tag{33}$$

Therefore the integral

$$\int_{-\infty}^{\infty} \rho dx = \text{constant} \tag{35}$$

represents the conservation of the total mass in the system.

Example

The conservation of electric charge is expressed by

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{36}$$

where $\rho(\mathbf{x}, t)$ is the charge density and $\mathbf{j}$ is the density current.

It follows that

$$\begin{aligned} \frac{d}{dt} \iiint_\Omega \rho(\mathbf{x}, t) d\mathbf{x} &= \iiint_\Omega \frac{\partial \rho(\mathbf{x}, t)}{\partial t} \, d\mathbf{x} \\ &= - \iiint_\Omega \nabla \cdot \mathbf{j} \, d\mathbf{x} - \iint_{\partial\Omega} \mathbf{j} \cdot \hat{\mathbf{n}} \, dS = 0. \end{aligned} \tag{37}$$

This implies that the total charge

$$\iiint_\Omega \rho(\mathbf{x}, t) d\mathbf{x} = \text{constant}. \tag{38}$$

A general form of 1-dimensional ***conservation law*** is

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \qquad (40)$$

where $T(x, t, u, u_x, \ldots)$ is the ***density*** and $X(x, t, u, u_x, \ldots)$ is the ***flux***

N.B. They ***cannot*** depend on u_t.

T and X_x are integrable and

$$X \to \text{constant as } |x| \to \infty. \qquad (41)$$

The conservation law implies

$$\int_{-\infty}^{\infty} T dx = \text{constant.} \quad (43)$$

The integral of T is ***conserved****, or is a* ***constant of motion****.*

The KdV equation

Consider the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0.$$

It is already in the form of conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u_{xx} - 3u^2\right) = 0. \qquad (45)$$

The density and the flux are given respectively by

$$T(x,t) = u(x,t) \tag{46a}$$

$$X(x,t) = u_{xx} - 3u^2. \tag{46b}$$

Therefore, we have

$$\int_{-\infty}^{\infty} u \, dx = \text{constant.} \tag{48}$$

The above integral expresses the ***mass conservation.***

If we multiply the KdV equation by u we obtain another conservation law

$$\frac{\partial\left(\frac{1}{2}u^2\right)}{\partial t} + \frac{\partial}{\partial x}\left(uu_{xx} - \frac{1}{2}u_x^2 - 2u^3\right) = 0. \qquad (50)$$

Therefore we have

$$\int_{-\infty}^{\infty} u^2 \, dx = \text{constant.} \quad (52)$$

The above integral expresses ***momentum conservation.***

We now construct the density for energy conservation.

Consider

$$3u^2 \times (\text{KdV}) + u_x \times \frac{\partial}{\partial x}(\text{KdV}). \tag{53}$$

This gives

$$3u^2 \left(u_t - 6uu_x + u_{xxx}\right) + u_x \left(u_{xt} - 6u_x^2 - 6uu_{xx} + u_{xxxx}\right) = 0, \tag{54}$$

which can be written as

$$\frac{\partial}{\partial t}\left(u^3 + \frac{1}{2}u_x^2\right) + \frac{\partial}{\partial x}\left(-\frac{9}{2}u^4 + 3u^2 u_{xx} - 6uu_x^2 + u_x u_{xxx} - \frac{1}{2}u_{xx}^2\right) = 0. \tag{55}$$

The previous expression is in a form of conservation law. Therefore, we have

$$\int_{-\infty}^{\infty} \left(u^3 + \frac{1}{2} u_x^2 \right) \, dx = \text{constant.} \qquad (57)$$

This integral expresses the ***energy conservation*** for water waves.

In the 1960s 8 more integral of motion for he KdV were found by trial, ingenuity and error. What is their physical meaning?

Gardner found an infinity of conservation laws.

We introduce the *Gardner transformation.*

$$u = w + \epsilon^2 w + \epsilon w_x \qquad (59)$$

The KdV equation now becomes

$$
\begin{aligned}
u_t - 6uu_x + u_{xxx} &= w_t + \epsilon w_{xt} + 2\epsilon^2 w w_t - 6\left(w + \epsilon w_x + \epsilon^2 w^2\right) \\
&\quad \times \left(w_x + \epsilon w_{xx} + 2\epsilon^2 w w_x\right) + w_{xxx} + \epsilon w_{xxxx} + 2\epsilon^2 (w w_x)_{xx} \\
&= \left(1 + \epsilon \frac{\partial}{\partial x} + 2\epsilon^2 w\right) \left[w_t - 6\left(w + \epsilon^2 w^2\right) w_x + w_{xxx}\right]. \qquad (60)
\end{aligned}
$$

u is a solution of the KdV equation if w is a solution of

$$
\boxed{w_t - 6(w + \epsilon^2 w^2) w_x + w_{xxx} = 0}
$$

NB The opposite ***is not*** necessarily true!

The previous equation can be rewritten as

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(w_{xx} - 3w^2 - 2\epsilon^2 w^3\right) \quad (62)$$

and therefore

$$\int_{-\infty}^{\infty} w \; dx = \text{constant.} \quad (64)$$

We now set

$$w(x,t;\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n w_n(x,t), \quad \epsilon \to 0. \qquad (66)$$

This series is asymptotic and does not need to be convergent. Then

$$\int_{-\infty}^{\infty} w_n \, dx = \text{constant}, \quad n = 0, 1, \ldots \qquad (68)$$

- We insert the asymptotic expansion for w in the Gardner transformation;
- We equates coefficients of ϵ^n for each $n = 0, 1, \ldots$.

$$\sum_{n=0}^{\infty} \epsilon^n w_n \sim u - \epsilon \sum_{n=0}^{\infty} \epsilon^n w_{nx} - \epsilon^2 \left(\sum_{n=0}^{\infty} \epsilon^n w_n \right)^2 \qquad (70)$$

$$
\begin{aligned}
w_0 &= u; \quad w_1 = -w_{0x} = -u_x; \quad w_2 = -w_{1x} - w_0^2 = u_{xx} - u^2 \\
w_3 &= -w_{2x} - 2w_0 w_1 = -(u_{xx} - u^2)_x + 2uu_x; \\
w_4 &= -w_{3x} - 2w_0 w_2 - w_1^2 \\
&= -\left\{2uu_x - (u_{xx} - u^2)_x\right\}_x - 2u(u_{xx} - u^2) - u_x^2 \\
&\dots
\end{aligned}
\tag{72}
$$

The integrals of exact differentials (in x) are zero.

We have

$$w_{2m+1} = \frac{\partial}{\partial x}\,(\text{something})\,. \tag{73}$$

Therefore trivially

$$\int_{-\infty}^{\infty} w_{2m+1}\,dx = 0 \tag{74}$$

All the non trivial constants of motion are given by

$$\int_{-\infty}^{\infty} w_{2m}\,dx = \text{constant}, \quad m = 0, 1, \ldots \tag{76}$$

w_0, w_2 and w_4 are the constants of motions previously found:

$$\int_{-\infty}^{\infty} w_0 \, dx = \int_{-\infty}^{\infty} u \, dx = \text{constant}, \tag{80}$$

$$\int_{-\infty}^{\infty} w_2 \, dx = \int_{-\infty}^{\infty} u^2 \, dx = \text{constant}, \tag{81}$$

$$\int_{-\infty}^{\infty} w_4 \, dx = \int_{-\infty}^{\infty} \left(u^3 + \frac{1}{2} u_x^2 \right) \, dx = \text{constant}. \tag{82}$$

6. The Lax pair

Question: *Is the KdV equation the only evolution equation with the special properties that we have studied? (i.e. solution by inverse scattering theory,* ***constancy*** *of the eigenvalues associated to the direct scattering problem and therefore simple time-evolution of the scattering data.)*

Lax (1968) found a reason underlying the fact that the eigenvalues of the scattering problem are constant while the solution of the KdV equation evolves in a complicated way. The KdV equation does not stand alone in the class of evolution equations.

Suppose that we want to solve the initial value problem given by

$$u_t = N(u)$$

with $u(x,0) = f(x)$, with $u \in Y$ (Y being some appropriate function space) and $N : Y \to Y$ is a operator independent of t but in general depending on u, x, and on the derivatives of u w.r.t. x.

N ***need not be*** a partial differential operator.

An Hilbert space $\mathcal{H}$ is a vector linear space (possibly infinite dimensional), ***complete****, i.e. each element $\psi \in \mathcal{H}$ can be expressed as*

$$\psi = \sum_{n=1}^{\infty} c_n \phi_n, \tag{84}$$

where c_n are constants and $\{\phi_n\}$ is an appropriate basis. $\mathcal{H}$ must also be equipped with a scalar product (ϕ, ψ).

A scalar product is a bilinear function $\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ *with the following properties*

$$(\psi, \phi) = (\phi, \psi)^* \tag{89}$$

$$(\psi, \phi + \xi) = (\psi, \phi) + (\psi, \xi) \tag{90}$$

$$(0, \psi) = 0 \tag{91}$$

$$(\psi, \psi) \geq 0, \quad (\psi, \psi) = 0 \quad \textit{iff}\ \psi = 0. \tag{92}$$

Def.: *A linear operator* L *in* $\mathcal{H}$ *is self-adjoint if* $(\phi, L\psi) = (L\phi, \psi)$. *All the eigenvalues of* L *are real.*

Examples

(i) Any finite-dimensional vector space with the usual scalar product is an Hilbert space. Linear operators are given by matrices.

(ii)The space of square-integrable functions in the interval $[0, 2\pi)$, $L^2([0, 2\pi))$, is an Hilbert space. Each function $f(x)$ can be expanded in a Fourier series

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{93}$$

The scalar product is defined by

$$\int_0^{2\pi} g^*(x) f(x) dx. \tag{94}$$

In our case $N(u)$ is the KdV equation

$$N(u) = 6uu_x - u_{xxx}$$

We now suppose that the evolution equation can be expressed as

$$L_t = ML - LM,$$

i.e. $u_t - N(u) = L_t + LM - ML = L_t + [L, M] = 0.$

*M and L are **linear** operator acting on a Hilbert space $\mathcal{H}$ and which may depend on $u(x,t)$. We also assume that L is self-adjoint.*

We now introduce the ***spectral*** equation

$$L\psi = \lambda\psi \text{ for } t \geq 0 \text{ and } -\infty < x < \infty.$$

Because L depends upon t, in general the eigenvalues $\lambda(t)$ depend upon t too.

By differentiating w.r.t. t the spectral equation, we have

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t. \tag{95}$$

By using $L_t = ML - LM$ the previous equation becomes

$$\begin{aligned} \lambda_t \psi &= (L-\lambda)\psi_t + (ML - LM)\psi \\ &= (L-\lambda)\psi_t + M\lambda\psi - LM\psi \\ &= (L-\lambda)(\psi_t - M\psi). \end{aligned} \tag{96}$$

By taking the scalar product with ψ of the above expression, we have

$$\begin{aligned} (\psi,\psi)\lambda_t &= (\psi, (L-\lambda)(\psi_t - M\psi)) \\ &= (((L-\lambda)\psi, (\psi_t - M\psi)) \\ &= (0, (\psi_t - M\psi)) = 0. \end{aligned} \tag{97}$$

($L - \lambda$ is self-adjoint.)

It follows that $\lambda_t = 0$ and therefore

$$\lambda = \text{constant.}$$

From eq. (96) with $\lambda_t = 0$ it also follows that

$$(L - \lambda)(\psi_t - M\psi) = 0.$$

$(\psi_t - M\psi)$ is therefore an eigenvector of L with eigenvalue λ. Therefore, we have

$$\psi_t - M\psi = \alpha(t)\psi$$

where $\alpha(t)$ is a scalar function of t.

We now set

$$M' = M + \alpha(t)I,$$

where I is the identity operator.

Since L commutes with $\alpha(t)I$, from the equation $L_t = ML - LM$ it follows that

$$L_t = M'L - LM'.$$

We can therefore redefine M by using M': this ***will not*** alter $L_t = ML - LM$ (which is a representation of $u_t - N(u)$).

We therefore have the ***time-evolution equation*** for ψ

$$\psi_t = M\psi.$$

These results can be summarized in the following

Theorem: *If the evolution equation*

$$u_t - N(u) = 0 \tag{101}$$

can be expressed as the ***Lax equation***

$$L_t + LM - ML = L_t + [L, M] = 0 \tag{102}$$

and if $L\psi = \lambda\psi$, then $\lambda_t = 0$ and ψ evolves according to

$$\psi_t = M\psi. \tag{103}$$

6.1 The Lax KdV hierarchy

We now want to apply the previous idea to the KdV equation, therefore

$$N(u) = 6uu_x - u_{xxx}$$

The problem is ***How to choose*** L ***and*** M ***?***

In our case the obvious choice for L is the Schrödinger operator

$$L = -\frac{\partial^2}{\partial x^2} + u(x,t) \qquad (105)$$

It turns out that M must be a skew-symmetric operator

$$(\phi, M\psi) = -(M\phi, \psi)$$

A natural choice is therefore to construct M from a suitable linear combination of ***odd*** derivatives.

Consider the inner product $(\phi, \psi) = \int_{-\infty}^{\infty} \phi\psi dx$, then

$$(M\phi, \psi) = \int_{-\infty}^{\infty} \frac{\partial^n \phi}{\partial x^n} \psi dx = -\int_{-\infty}^{\infty} \phi \frac{\partial^n \psi}{\partial x^n} dx = -(\phi, M\psi). \quad (106)$$

Moreover, $L_t + [L, M]$ must be a multiplicative operator.

Consider the simplest choice

$$M = c\frac{\partial}{\partial x} \qquad (108)$$

for c constant, then

$$[L, M] = c\left(-\frac{\partial^2}{\partial x^2} + u(x,t)\right)\frac{\partial}{\partial x} - c\frac{\partial}{\partial x}\left(-\frac{\partial^2}{\partial x^2} + u(x,t)\right) = -cu_x \qquad (109)$$

Note that if $\partial/\partial x$ and $a(x)$ are two operators, their composition the $\partial/\partial x[a(x)]$ applied to $b(x)$ gives

$$\frac{\partial}{\partial x}\left(a(x)b(x)\right) = a_x b + a b_x \qquad (110)$$

Finally, we have

$$L_t + [L, M] = u_t - cu_x.$$

The one-dimensional wave equation $u_t - cu_x$ has an associated spectral problem with eigenvalues which are ***constant of motion.***

We now choose M so that it involves at most a third-order differential operator,

$$M = -\alpha \frac{\partial^3}{\partial x^3} + U \frac{\partial}{\partial x} + \frac{\partial}{\partial x} U + A. \tag{112}$$

Here α is a constant, $U = U(x,t)$ and $A = A(x,t)$.

After simple algebra, we have

$$\begin{aligned} [L, M] = {} & \alpha u_{xxx} - U_{xxx} - A_{xx} - 2u_x U \\ & + (3\alpha u_{xx} - 4U_{xx} - 2A_x) \frac{\partial}{\partial x} + (3\alpha u_x - 4U_x) \frac{\partial^2}{\partial x^2} \end{aligned} \tag{113}$$

It follows that $[L, M]$ is a multiplicative operator if

$$U = \frac{3}{4}\alpha u \quad \text{and} \quad A = A(t) \tag{115}$$

The Lax equation now becomes

$$L_t + [L, M] = u_t - \frac{3}{2}\alpha u u_x + \frac{1}{4}\alpha u_{xxx} = 0. \tag{117}$$

For $\alpha = 4$ we recover the KdV equation. The operator M now becomes

$$M = -4\frac{\partial^3}{\partial x^3} + 3u\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}u + A(t) \qquad (119)$$

and so the time-evolution equation for ψ is

$$\psi_t = -4\psi_{xxx} + 3u\psi_x + 3\left(u\psi\right)_x + A\psi. \qquad (120)$$

The previous equation can be recast, on using the Schrödinger equation

$$\psi_{xx} + (\lambda - u)\,\psi = 0, \tag{121}$$

as

$$\begin{aligned} \psi_t &= 4\,(\lambda\psi - u\psi)_x + 3u\psi_x + 3\,(u\psi)_x + A\psi \\ &= 2\,(u + 2\lambda)\,\psi_x - u_x\psi + A\psi \end{aligned} \tag{122}$$

For $A = 0$, this is the time-evolution equation that we found for the discrete eigenfunctions; with $A = 4ik^3$ we have the corresponding equation for the equation for the continuous eigenfunctions.

The KdV equation is the second example in the Lax-formulation framework with L being the Scrödinger operator. The procedure adopted can be extended to higher-order nonlinear evolution equations.

Trial and error leads to

$$M = -\alpha \frac{\partial^{2n+1}}{\partial x^{2n+1}} + \sum_{m=1}^{n} \left(U_m \frac{\partial^{2m-1}}{\partial x^{2m-1}} + \frac{\partial^{2m-1}}{\partial x^{2m-1}} U_m \right) + A \quad (124)$$

where α is a constant, $U_m = U_m(x, t)$ and $A = A(t)$.

The restriction that $[L, M]$ must be a multiplicative operator imposes n conditions on the n unknown function U_m.

$n = 1$ gives the KdV equation. It can be shown that for $n = 2$ the evolution equation is

$$u_t + 30u^2 u_x - 20u_x u_{xx} - 10uu_{xxx} + u_{xxxxx} = 0. \quad (126)$$