ABSTRACT

Here we present an investigation into how cooling of the plasma influences the oscillation properties (e.g. eigenfunctions and eigenfrequencies) of transverse (i.e. kink) magnetohydrodynamic (MHD) waves in a compressible magnetic flux tube embedded in a gravitationally stratified and uniformly magnetised atmosphere. The cooling is introduced via a temperature dependent density profile. A time dependent governing equation is derived and an approximate zeroth order solution is then obtained. From this the influence of cooling on the behaviour of the eigenfrequencies and eigenfunctions of the transverse MHD waves is determined for representative cooling time scales. It is shown analytically, as the loop cools, how the amplitude of the perturbations is found to decrease as time increases. For cooling timescales of 900 – 2000 s (as observed in typical EUV loops) it is shown that the cooling has important and relevant influence on the damping times of loop oscillations. Next, the theory is put to the test. The damping due to cooling is fitted to a representative observation of standing kink oscillation of EUV loops. It is also shown with an explicit approximate analytical form, how the period of the fundamental and first harmonic of the kink mode changes with time as the loop cools. A consequence of this is that the value of the period ratio $P_1/P_2$, a tool that is popular in magneto-seismological studies in coronal diagnostics, decreases from the value of a uniform loop, 2, as the temperature decreases. The rate of change in $P_1/P_2$ is dependent upon the cooling time scale and is well within the observable range for typical EUV loops. Further to this, the magnitude of the anti-node shift of the eigenfunctions of the first harmonic is shown to continually increase as the loop cools, giving additional impetus to the use of spatial magneto-seismology of the solar atmosphere. Finally, we suggest that measurements of the rate of change in the eigenfunctions and eigenfrequencies of MHD oscillations can provide values for the cooling timescale and a further insight into the physics of coronal loops.

Subject headings: magnetohydrodynamics(MHD) - plasmas - Sun: corona - waves

1. INTRODUCTION

Detailed observations have revealed that the plasma fine structure of the solar corona is of a highly complex, magnetised and a rather dynamic nature (see, e.g. Brooks et al., 2007; Mariska et al., 2007). The coronal plasma is often characterised by clearly delineated loops that outline the coronal magnetic field and are undergoing continual evolution (Schrijver et al., 1999). The study of coronal loops is of much interest as it is widely thought that they may provide an answer to the problem of coronal heating because of their role as building blocks of the magnetised solar atmosphere (for recent reviews see, e.g. Erdélyi & Ballai, 2007; Erdélyi, 2008; Taroyan, 2008; Taroyan & Erdélyi, 2008).

Interest in the nature of coronal loops was generated by their discovery in the 1970’s by Skylab. However, it was not until the Transition Region and Coronal Explorer (TRACE) was launched that the highly complex structure of the loops was revealed. TRACE also enabled the first direct observations of coronal loops undergoing heavily damped periodic motion. Theoretical work investigating the magneto-acoustic oscillations of a straight, homogeneous magnetic flux tube by Edwin & Roberts (1983) made interpretation of these observed periodic events possible. Edwin & Roberts (1983) showed that the magnetic flux tube is able to support a large number of different magneto-acoustic modes and the properties of these modes (i.e. phase speed and period) were found to be dependent upon the local plasma parameters (e.g. density, magnetic field strength).

However, the model of Edwin & Roberts (1983) is a simplified version of the true nature of the plasma fine structure. The limitations of this model are highlighted by observations of loops undergoing multimode standing oscillations (see Verwichte et al., 2004; van Doorsselaere et al., 2007; Verth et al., 2008). The transversal motion of the loops have been interpreted as the fast kink mode, where both the fundamental and first harmonic were found to be present. The ratio of the periods of these two modes, $P_1/P_2$, should be equal to 2 in a uniform loop. The observed values of the period ratio are, however, mostly $P_1/P_2 < 2$, e.g. Verth et al. (2008) observed $P_1/P_2 = 1.54$.

The concept of a uniform monolithic loop put forward by Edwin & Roberts (1983) has recently been improved by a number of authors to include various geometrical and physical (often second order) effects, e.g. gravitational or magnetic stratification, structuring, loop expansion, damping, cross-sectional shape, loop geometry, stage of loop emergence, wave leakage, magnetic twist, loop curvature, to see how they affect the propagation of the various wave modes supported within the flux tube. One important outcome of these theoretical investiga-
tions is the suitability of solar magneto-seismology, or coronal seismology when applied to coronal loops. For a recent comprehensive review on kink oscillations see Ruderman & Erdélyi (2009) and Andries et al. (2009) for specific coronal applications.

It was suggested by Roberts et al. (1984) that measured properties of the MHD oscillations within local coronal structures (e.g. loops) may be inverted to obtain information about otherwise unmeasurable or hardly measurable solar atmospheric parameters. After the observations by Verwichte et al. (2004), it was pointed out that the period ratio, $P_1/P_2$, could provide information on the density scale height within the corona (Andries et al., 2005a). It has been shown that, so far, there are two main effects that influence the value of the period ratio. One effect is longitudinal density structuring (Andries et al., 2005a; McEwan et al., 2006) and the other is the longitudinal variation of the magnetic field (Verth & Erdélyi, 2008; Ruderman et al., 2008). However, further studies have highlighted that a variety of other parameters also need to be taken into account when determining the density scale height from measurements of the period ratio, e.g. finite tube width (McEwan et al., 2006) and loop geometry (Andries et al., 2005a; Dymova & Ruderman, 2006; Verth et al., 2008; Erdélyi & Morton, 2009, Morton & Erdélyi, 2009).

Further to this, Andries et al. (2005b), Erdélyi & Verth (2007) and Verth et al. (2007) have shown that density stratification also causes a measurable change in the amplitude profile of transverse oscillations in coronal loops, with these studies laying down the foundations of spatio-magneto-seismology. The result of most significance was that the anti-nodes of the first harmonic were found to undergo a spatial shift due to the density stratification. They then suggested that measurement of this anti-node shift could provide information on the density profile of the loop.

All of these previous studies assumed that coronal loops have a time independent background, i.e. equilibrium quantities (e.g. temperature, density) do not change on time scales comparable to the characteristic period of oscillations. However, this assumption is often not satisfied for oscillating loops. One dynamic feature that should certainly be considered when interpreting MHD oscillations is the temperature evolution of coronal loops. The temporal evolution of loop temperature has been observationally investigated by many authors, e.g. Winebarger et al. (2003); Nagata et al. (2003); López Fuentes et al. (2007). Nagata et al. (2003) observed that there are at least two types of temperature evolution. The first is that of hot loops ($T > 2.5$ MK - soft x-ray temperatures) which appeared first in the Yohkoh Soft X-ray Telescope (SXT) and then seen to cool down into the EUV temperature range ($1 - 2$ MK) and become visible to the Extreme-ultraviolet Imaging Telescope (EIT) onboard the Solar Heliospheric Observatory (SOHO). This process has also been observed by Winebarger et al. (2003), Winebarger & Warren (2005). Winebarger & Warren (2005) identified five loops with $T > 2.5$ MK in SXT and followed their evolution into the EUV temperature range with TRACE. The time delay between the appearance of the loops in SXT and their appearance in the TRACE filters can be used to calculate an estimate for the cooling time. Using a temperature profile exponentially decreasing in time, cooling times of 4000 – 12000 s can be obtained. It has been observed that these cooling loops are mainly situated in the center of active regions (see, e.g. Ugarte-Urra et al., 2009).

The second form of temperature evolution is that of long lived cool loops. The loops are seen in EIT images on the periphery of active regions, lasting for a duration of around 16 hours with temperatures in the range 0.4 – 1.3 MK. No brightening is seen in SXT before they appear in EIT. Further to this, López Fuentes et al. (2007) studied the long term evolution of coronal loop using the Geosynchronous Operational Environmental Satellite 12 (GOES-12) Solar X-ray Imager (SXI). They observed that the evolution of the loops could be separated into three phases: (i) the rise phase, where loop intensities increases due to a slowly increasing heating rate; (ii) main phase, where the loop intensity increases/decreases at a lower rate; (iii) the decay phase, where the intensity decreases and cooling dominates the heating.

There has been a number of theoretical investigations into the methods of heating (see, e.g. Aschwanden et al., 2001; Mendoza-Bricéo et al., 2002; Taroyan et al., 2006; Taroyan & Erdélyi, 2008; Bradshaw & Cargill, 2006) and cooling (see, e.g. Cargill, 1994; Bradshaw & Cargill, 2005; Bradshaw, 2008) of coronal loops. However, the cooling process of a loop does not appear to follow a single description. It has been observed that some loops undergo a rapid cooling process, referred to as catastrophic cooling, where the loop cools significantly within a couple of hundred seconds (see, e.g. Levine & Withbroe, 1977; Schrijver, 2001; Mendoza-Bricéo et al., 2005). This is in contrast to the cooling times observed on the scale from 10 minutes to hours, e.g. Winebarger et al. (2003), Winebarger & Warren (2005); Aschwanden & Terradas (2008).

In this paper we investigate whether transverse MHD oscillations in coronal loops that are in the decay phase, i.e. the cooling phase, are significantly influenced by the change in temperature. A micro-physical process of cooling is not specified as it is still unknown which mechanism (either radiation or conduction) has the strongest influence on the transverse oscillations. It has been suggested by the initial numerical studies of Aschwanden & Terradas (2008) that radiative cooling will be the dominant method for EUV loops, where $T \sim 1 - 2$ MK. For high temperatures, $T > 5$ MK, thermal conduction is expected to dominate the cooling process. For longitudinal waves, cooling by thermal conduction has been found to provide a method of damping that can account for the
rapid damping seen in observations (see, e.g. Ofman & Wang, 2002; Mendoza-Briceno et al., 2004; Erdélyi et al., 2008; de Moortel, 2009).

First, a solution to the ideal MHD equations is sought and a time dependent governing equation for wave motion within a magnetic cylinder is obtained. An approximate solution of this equation is then found using the WKB method. Inspired by Verth et al. (2008), who used magneto-seismology (in particular, the period ratio) to offer an explanation for the dimming in terms of cooling, we then investigate the effect of cooling on the eigenfrequencies and eigenfunctions, i.e. the two most popular tools of magneto-seismology. We find explicitly that the period ratio $P_1/P_2$ of the loop decreases from its initial value as the temperature decreases. This behaviour is somewhat expected, since while the loop cools it becomes lighter and their eigenfrequencies are most likely higher. A continually increasing anti-node shift also occurs as the loop cools. Further to this, the cooling of the loop also provides a very elegant explanation of damping perturbations.

2. GOVERNING EQUATION FOR TIME DEPENDENT MHD OSCILLATIONS

2.1. Background configuration

In order to study the effect of cooling on transverse magneto-acoustic oscillations we model the cooling of coronal loops with an exponentially decaying temperature profile of the form

$$T(t) = T_0 \exp \left\{ -\frac{(t - t_0)}{\tau_{cool}} \right\},$$

where $T_0$ is the initial temperature, $t_0$ is the time at which the cooling begins and $\tau_{cool}$ is the cooling time. This profile is found to be a good fit for loops cooling between the range 1.5 – 3 MK (see, e.g. Ugarte-Urra et al., 2009) and was implemented in the numerical studies of Aschwanden & Terradas (2008). It also is assumed here that the temperature profile along the loop is isothermal. The isothermal nature of coronal loops is still much debated, there is observational evidence for many different temperature profiles within the solar corona (see, e.g. Aschwanden & Nightingale, 2005; Schmelz et al., 2009). Because a time dependent temperature profile has been introduced it will be necessary to find a time dependent solution to the background MHD equations that will be perturbed linearly by transversal oscillations.

To model the effect of cooling, the MHD equations will be simplified by ignoring the direct terms related to the various cooling processes (e.g. conduction, radiation), instead introducing the temperature evolution via the hydrodynamic relationship between the density scale height and temperature,

$$H(t) = 47T(t) \left[ \frac{Mn}{MK} \right].$$

Here $T(t)$ is measured in MK. This means the cooling will be introduced via its influence on the density profile. Hence, a density profile dependent upon the scale height is required. Now, let

$$\rho_0 = \rho_f \exp \left\{ -\frac{h_s}{H(t)} \cos \left( \frac{\pi z}{2L} \right) \right\},$$

where $h_s$ is the height of the loop apex above the photosphere and $\rho_f$ is the density at the footpoints of a coronal loop. A density profile of this form describes a semi-circular loop embedded in a gravitationally stratified atmosphere. It should be noted here that recent observations using STEREO by Aschwanden et al. (2008) show that the majority of loops are in fact non-circular in shape. An elliptical shape would provide a better approximation but adds further complication to the equations (see Morton & Erdélyi, 2009) and would obscure the main point. A circular shape is still a good first approximation.

The background state is described by

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{U}) = 0,$$

$$\rho_0 \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + \nabla p_0 = 0,$$

$$\left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) e + (\gamma - 1) e \nabla \cdot \mathbf{U} = \mathbf{L},$$

where $e \approx c_s T$ is the internal energy, $\gamma = c_p/c_v$ is the ratio of specific heats at constant pressure and volume, $L$ is some mechanism for cooling the loop and $\mathbf{U} = (0,0,U_z) = U_z(z,t)$ is the background flow. For the density profile given by Eq. (3), it can be obtained from the background equations $U_z \propto L/\tau_{cool}$. The equation of mass conservation implies that as the loop cools, the mass at the loop apex flows down and flows out of the loop at the footpoints.

Consider a straight magnetic flux tube with a cross-section of radius $R$ and half length $L$ which is embedded in a magnetic environment (see Fig. 1). The magnetic field inside and outside the tube is uniform, with the internal (external) magnetic field of the form $\mathbf{B}_i = B_i \hat{z}$ ($\mathbf{B}_o = B_o \hat{z}$). Here, $\hat{z}$ is the unit vector in the $z$-direction and it is assumed $B_i \approx B_o$ for coronal conditions. Linear perturbations about the dynamic background for this model are described by the linear, ideal MHD equations

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{v}) + \nabla \cdot (\rho_1 \mathbf{U}) = 0,$$

$$\rho_0 \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{U} \right) = -\nabla P_T + \frac{B_o}{\mu_0} \frac{\partial \mathbf{b}}{\partial z},$$

$$\frac{\partial \rho_1}{\partial t} + \mathbf{U} \cdot \nabla \rho_1 + \mathbf{v} \cdot \nabla \rho_0 = 0,$$

$$c_s^2 \left( \frac{\partial \rho_1}{\partial t} + \mathbf{U} \cdot \nabla \rho_1 + \mathbf{v} \cdot \nabla \rho_0 \right),$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{b}) + \nabla \times (\mathbf{v} \times \mathbf{B_0}).$$

Here $p_0$ is the background pressure, $p_1$ is the pressure perturbation, $\rho_1$ is the density perturbation, $\mathbf{v}$ is the velocity perturbation, $\mathbf{B}_0$ is the background magnetic field, $\mathbf{b}$ is the perturbation of the magnetic field, $c_s^2 = \gamma p_0/\rho_0$ is the sound speed of the plasma and $\mu_0$ is the magnetic permeability of free space.

The perturbation of the total pressure is $P_T = p_1 + p_{m}$ where the perturbation of the magnetic pressure is given by

$$p_{m} = \frac{B_0 d_z}{\mu_0}.$$
Now, some assumptions are made about the coronal plasma. First, we make the assumption that $U$ is small, i.e. $U \ll v_A$, where $v_A^2(z, t) = B_0^2/\rho_0(z, t)\mu_0$. This approximation is likely to be valid in the corona as observed flows along coronal loops have a wide range of velocities from $40 - 120 \text{ km s}^{-1}$ (see e.g. Schrijver et al., 1999; Ofman & Wang, 2008), whereas typical Alfvén velocities in the coronal loops are between $1000 - 5500 \text{ km s}^{-1}$ (see Aschwanden et al., 2002). Second, we make the assumption that the plasma beta, given by $\beta = 2\mu_0\rho_0/B_0^2$, in the corona is small. This approximation is valid as $\beta \sim 0.01$ is the characteristic value in the corona, hence $c_s \ll v_A$. Equipped with the above assumptions we now concentrate on the transverse oscillations and can ignore the $v_r$ component.

Using these assumptions, an equation describing the evolution of the total pressure perturbation is obtained from the ideal MHD equations (see Appendix A), namely

$$\frac{\partial^2 P_r}{\partial z^2} = v_A^2(z, t)\nabla^2 P_r. \quad (6)$$

Now, taking the $r$ component of Eq. (5e), namely

$$\frac{\partial b_r}{\partial t} = B_0 \frac{\partial (v_r)}{\partial z} - U_z \frac{\partial (b_r)}{\partial z} - b_r \frac{\partial U_z}{\partial z}, \quad (7)$$

and then differentiating with respect to $z$,

$$\frac{\partial^2 b_r}{\partial t^2} = B_0 \frac{\partial^2 (v_r)}{\partial z^2} - 2U_z \frac{\partial (b_r)}{\partial z} - b_r \frac{\partial^2 U_z}{\partial z^2} - U_z \frac{\partial^2 b_r}{\partial z^2}. \quad (8)$$

Then substituting in the $r$ component of Eq. (5b), we obtain

$$\frac{1}{v_A^2(z, t)} \frac{\partial^2 v_r}{\partial t^2} = \frac{\partial^2 v_r}{\partial z^2} - \frac{\mu_0 \partial \rho_0}{B_0^2} \left( \frac{\partial v_r}{\partial t} + U_z \frac{\partial v_r}{\partial z} \right) - \frac{2}{v_A^2(z, t)} \frac{\partial^2 U_z}{\partial z} \frac{\partial v_r}{\partial z} - \frac{\mu_0 U_z}{B_0^2} \frac{\partial P_T}{\partial t} - \frac{2\mu_0 U_z}{B_0^2} \frac{\partial P_T}{\partial z} - \frac{b_r}{B_0^2} \frac{\partial^2 U}{\partial z^2}. \quad (9)$$

At the boundary of the tube, located at $r = R$, the equilibrium plasma quantities change discontinuously. The dynamic boundary conditions that are required to be satisfied are the continuity of total pressure, $P_T$, and radial velocity. The velocity perturbation can be written in terms of the Lagrangian displacement, $\xi_r$, as such

$$v_r = \frac{\partial \xi_r}{\partial t} + U \frac{\partial \xi_r}{\partial z}. \quad (10)$$

Across the boundary $R$ we then require that

$$v_{r1} - v_{r2} = (U_i - U_e) \frac{\partial \xi_r}{\partial z}, \quad (11)$$

is satisfied. As a small flow has been assumed both inside and outside the loop, the further assumption is now made that $U_i \approx U_e$. To obtain continuity at the tube boundary, it then follows that the jump conditions

$$[v_r] = 0, \quad [P_T] = 0, \quad [b_r] = 0 \quad \text{are satisfied.}$$

The frozen in condition also provides boundary conditions at $z = \pm L$, namely

$$v_r = v_0 = 0 \quad \text{at} \quad z = \pm L. \quad (13)$$

Now we introduce the first dimensionless parameter $\epsilon = R/L$. For coronal loops it is observed that $R \ll L$, e.g. typical values are $R \approx 6 \text{ Mm}$, $L \approx 100 \text{ Mm}$, hence $\epsilon$ is a small quantity. The introduction of this small parameter suggests the use of stretched coordinates. Consequently the system of stretching coordinates are $\zeta = \epsilon z$ and $\xi_r = r\epsilon$ outside the tube and $\zeta = z$ inside. This essentially reduces the ideal MHD equations to describing a one-dimensional loop model in the thin flux tube approximation.

### 2.2. Solution for the internal and external regions

After changing to the new system of coordinates for the internal region of the tube, the perturbed quantities can then be Fourier analysed in the angular direction, i.e. take the perturbed quantities proportional to $\exp(i\theta)$. Neglecting terms of order $\epsilon^2$ in Eq. (6) gives

$$\frac{\partial}{\partial r} \left( r \frac{\partial P_{Ti}}{\partial r} \right) - \frac{m^2}{r} P_{Ti} = 0, \quad (14)$$
which has solution regular at \( r = 0 \) given by

\[
P_{T1} = C(\xi, t) \left( \frac{r}{R} \right)^m ,
\]

where \( P_{T1} \) is the total pressure inside the tube, \( C(\xi, t) \) is an arbitrary function of \( \xi \) and \( t \), and \( m \) is the azimuthal wavenumber with integer value.

Changing to the new system of coordinates for the external region of the tube, again Fourier analysing in the angular direction, Eq. (6) gives

\[
\frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \frac{\partial F}{\partial \xi} \right) + \frac{\partial^2 F}{\partial \xi^2} - \frac{m^2}{\xi^2} F - \frac{1}{\epsilon^2 v_x^2(\xi, t)} \frac{\partial^2 P_{Te}}{\partial t^2} = 0 ,
\]

where \( P_{Te} \) is the total pressure outside the tube. Notice that the time derivative does not appear in Eq. (14). This occurs because the time derivative term is much smaller than the \( r \) derivative terms.

Let \( P_{Te} = G(\xi, t) F(\xi) \). Eq. (16) can be separated to give

\[
1 \frac{\partial}{\partial \xi} \left( \xi \frac{\partial F}{\partial \xi} \right) - \frac{m^2}{\xi^2} F = k^2 F ,
\]

and

\[
\frac{1}{\epsilon^2 v_x^2} \frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial \xi^2} = k^2 G ,
\]

where \( k^2 \) is the separation constant. Of these two equations, Eq. (18) proves to be difficult to solve analytically. However, it turns out it is not necessary to find an analytic solution to this equation. In the external region we require perturbations to tend to zero as \( \xi \to \infty \). The solutions to Eq. (17) satisfying this condition are given by

\[
F(\xi) = \begin{cases} 
K_m(k\xi), & k^2 > 0, \\
\xi^{-m}, & k^2 = 0, \\
H_m^{(1)}(k\xi), & k^2 < 0,
\end{cases}
\]

where \( K_m(x) \) is the modified Bessel function of the second kind of order \( m \), \( H_m^{(1)}(x) \) is the first Hankel function of order \( m \) and \( k^2 \equiv -k^2 \). If the assumption \( \xi = \epsilon r \ll 1 \) is made then the asymptotic expansions for the Bessel and Hankel functions are

\[
K_m(x) \sim \frac{(m-1)!}{2} \left( \frac{x}{2} \right)^{-m} ,
\]

\[
H_m^{(1)}(x) \sim \frac{i(m-1)!}{\pi} \left( \frac{x}{2} \right)^{-m} .
\]

It can be seen from these expansions that \( F(\xi) \xi^{-m} \) for all values of the separation constant \( k^2 \). The solution to Eq. (16) can then be written in the form

\[
P_{Te} = A_m G(\xi, t) \left( \frac{r}{R} \right)^m ,
\]

where \( A_m \) is constant dependent upon \( k \) and \( m \). Now, using the jump condition given in Eq. (12), it is found from Eqs. (15) and (19) that \( C(\xi, t) = A_m G(\xi, t) \).

Using the internal and external solutions for the total pressure perturbation and Eq. (9), the governing equation is derived (see Appendix B), namely

\[
c_2^2(z, t) \frac{\partial^2 v_r(z, t)}{\partial z^2} - \frac{\partial^2 v_r(z, t)}{\partial t^2} = \frac{2p_i p_e}{(p_i + p_e)(p_i - p_e)} \left( \frac{1}{\rho_i} \frac{\partial p_i}{\partial t} - \frac{1}{\rho_e} \frac{\partial p_e}{\partial t} \right) \frac{\partial v_r}{\partial t} + c_2^2(z, t) b_r \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{\rho_i - p_e} \left( \frac{\partial p_i}{\partial t} - \frac{\partial p_e}{\partial t} \right) + 2 \frac{\partial U_z}{\partial z} \frac{\partial v_r}{\partial t} .
\]

Here,

\[
c_2^2(z, t) = \frac{2B^2}{\rho_0 (p_i(z, t) + p_e(z, t))},
\]

is the time dependent kink speed. Eq. (20) obeys the boundary conditions in Eq. (13) and is a generalisation of the eigenvalue equation found by Dymova & Ruderman (2005) and of that intuitively stated by Terradas et al. (2008). It can be seen that this equation still has a term dependent on \( b_r \), which, ideally, is unwanted. However, as will be seen in the following section, the term is not involved in the leading order equation.
3. Solution to Time Dependent Problem

To solve Eq. (20), the WKB method will be employed (see, e.g. Bender & Orszag, 1978). The Eqs. (1) and (2) can be combined to arrive at

\[ H(t) = 47T_0 \exp \left( \frac{t}{\tau_{cool}} \right) \left[ \frac{Mm}{Mk} \right]. \tag{21} \]

The rate of change of temperature for different values of cooling time is plotted for \( T_0 = 1.5 \) MK in Fig. 2. For the WKB method to be used here, it is required that there exists a small dimensionless parameter, \( \delta \). We now choose \( \delta = P/\tau_{cool} \), where \( P \approx 300 \) s is the characteristic period of the kink mode measured in the corona (see, e.g. Aschwanden et al., 1999). Then as long as \( \tau_{cool} \gg P \) then \( \delta \ll 1 \). It is then possible to identify the slow time scale given by \( T = \delta t \) and Eq. (21) can be re-written as

\[ H(t) = 47T_0 \exp \left( \frac{T}{P} \right) \left[ \frac{Mm}{Mk} \right]. \tag{22} \]

The slow time scale describes parameters that are slowly varying in time and only produce a non-negligible effect on the oscillations over an extended period of time, i.e. over the period \( \tau_{cool} \). It is worth commenting here that the accuracy of the solutions for the WKB method is dependent upon the size of \( \tau_{cool} \) compared to the characteristic period, \( P \). The larger the value \( \tau_{cool} \), i.e. the smaller \( \delta \), the greater the accuracy of the approximation.

Looking for a solution to the form

\[ v_\nu(z, T) = R \{ Q(z, T) \exp[i\delta^{-1} \Theta(z, T)] \}, \tag{23} \]

where \( R \) refers to the real part of a quantity, Eq. (20) can then be re-written

\[
c_k^2(z, T) \left\{ \frac{\partial^2 Q(z, T)}{\partial z^2} + \frac{i}{\delta} \left[ 2 \frac{\partial Q(z, T)}{\partial z} \frac{\partial \Theta(z, T)}{\partial z} + Q(z, T) \frac{\partial^2 \Theta(z, T)}{\partial z^2} \right] \right\}
+ Q(z, T) \frac{\partial^2 \Theta(z, T)}{\partial T^2} - i\delta \left[ 2 \frac{\partial Q(z, T)}{\partial T} \frac{\partial \Theta(z, T)}{\partial z} + Q(z, T) \right] \frac{\partial^2 \Theta(z, T)}{\partial T^2} \}
+ 2Q(z, T) \frac{\partial U_z}{\partial z} \frac{\partial \Theta(z, T)}{\partial T} \approx O(\delta^3). \tag{24} \]

where \( O(\delta^2) \) refers to terms of order \( \delta^2 \). It can be found from the background equations (4) that

\[
\frac{\partial U_z}{\partial z} \sim O(\delta), \quad \frac{\partial^2 U_z}{\partial z^2} \sim O(\delta) \]

The largest term in Eq. (24) is of order \( \delta^{-2} \), namely

\[
Q(z, T) \left( \frac{\partial \Theta(z, T)}{\partial z} \right)^2 = 0. \tag{25} \]

As \( \Theta(z, T) \) is an arbitrary function of \( z \) and \( T \) we choose \( \Theta(z, T) = \Theta(T) \), i.e. independent of \( z \), so that Eq. (25) is satisfied. This form of \( \Theta \) satisfies the equation of order \( \delta^{-1} \),

\[
2 \frac{\partial Q(z, T)}{\partial z} \frac{\partial \Theta(T)}{\partial z} + Q(z, T) \frac{\partial^2 \Theta(T)}{\partial z^2} = 0. \tag{26} \]

Now, separating out the zeroth (\( \delta^0 \)) and first order terms (\( \delta^1 \)), the zeroth order expression is given by

\[
\frac{\partial^2 Q(z, T)}{\partial z^2} + \frac{Q}{c_k^2(z, T)} \left( \frac{\partial \Theta(T)}{\partial T} \right)^2 = 0, \tag{27} \]

and the first order by

\[
2 \frac{\partial Q(z, T)}{\partial T} \frac{\partial \Theta(T)}{\partial T} + Q(z, T) \frac{\partial^2 \Theta(T)}{\partial T^2} = -2Q(z, T) \frac{\partial U_z}{\partial z} \frac{\partial \Theta(T)}{\partial T} - c_k^2(z, T) i b_r \frac{\partial^2 U_z}{\partial z^2}. \tag{28} \]

As mentioned previously, the term containing \( b_r \) only appears in the first order equation! A solution to Eq. (28) will not be attempted here as it is not needed to have an initial insight and is left to a later date.

We concentrate now on the zeroth order approximation. Because the function \( \Theta(T) \) is an arbitrary function of \( T \), we set it equal to

\[
\Theta(T) = \int \omega dT, \]

where \( \omega \) is the frequency of the oscillations, so that Eq. (27) can then be written

\[
\frac{\partial^2 Q(z, T)}{\partial z^2} + \frac{\omega^2}{c_k^2(z, T)} Q(z, T) = 0, \quad Q(z, T) = 0 \quad \text{at} \quad z = \pm L. \tag{29} \]

This is the time-dependent Sturm- Liouville problem and is also a generalisation of the governing equation found by Dymova & Ruderman (2005).

3.1. Analytical Solution

Eq. (29) provides a means of finding out how the cooling of the plasma affects the frequencies of the transverse oscillations. It would be ideal to find an analytical solution to (29), however, this proves to be a difficult task without making some further assumptions first and using an approximate method.

We then assume that \( \rho_i/\rho_e = \chi \), which means that \( \tau_{cool} \) is the same in both the internal and external plasma. This may not be the case in the corona and caution has to be exercised to avoid over interpretation of the results. Performing a Maclaurin expansion of the density profile, taking \( h_0/H \ll 1 \) and substituting into Eq. (29), the differential equation

\[
\frac{\partial^2 Q(z, t)}{\partial z^2} = -\frac{\omega^2}{c_k^2} \left\{ 1 - \frac{h_0}{H(t)} \cos \left( \frac{\pi z}{2L} \right) \right\} Q(z, t), \tag{30} \]

is obtained where \( c_k \) is the kink speed at the footpoints.

Now, implementing the variational method (see, e.g. McEvoy et al., 2008) to solve Eq. (30), we are able to obtain an approximate expression for the frequency of the fundamental kink mode

\[
\omega_1 \approx c_k \sqrt{\frac{\pi^3}{16L^2} \left[ \frac{\pi}{4} - \frac{2h_0}{3H(t)} \right]^{-1}}, \tag{31} \]

and for the frequency of the first harmonic of the kink mode

\[
\omega_2 \approx c_k \sqrt{\frac{\pi^3}{2L^2} \left[ \frac{\pi}{2} - \frac{16h_0}{15H(t)} \right]^{-1}}. \tag{32} \]
2. For the cooling loop, the scale height decreases as shown by these authors that as the value of the scale height is decreased, the ratio of the first harmonic of the kink mode. It was shown that magnetic structuring would increase physical effects may compete with each other. Let us recall that magnetic structuring would increase the frequency ratio (Verth & Erdélyi, 2008) so these two conditions. The values of the cooling time selected reflect the intermediate values of the cooling times that have been reported in EUV observations of coronal loops (Aschwanden & Terradas, 2008), however, the cooling time can be as low as 200 – 300 s (catastrophic cooling) or as large as 34,000 s (see Winebarger et al., 2003). As mentioned, the validity of the WKB method used to solve Eq. (20) becomes increasingly uncertain as $P/\tau_{cool} \rightarrow 1$, so we choose values of $\tau_{cool}$ large enough compared to a fixed representative period, to obtain a reasonable degree of accuracy. As for the opposite end of the spectrum, a large value of $\tau_{cool}$, again, compared to a fixed period, means that any cooling effect will be negligible over the time scale of oscillations. The typical temperature of EUV loops, as detected by e.g. TRACE, is 1.5 MK, we choose this to be the initial value of the temperature, i.e. $T_0$. We also select $h_0$ to be 38 Mm, corresponding to a typical loop length of 120 Mm.

We first plot the change in the value of the period for the fundamental kink mode, Fig. 3. The period is normalised with respect to the value of the period at $t = 0$, $P_i$. It can clearly be seen that the value of the period decreases as time increases and the rate of change of period

\[ \frac{\omega_2}{\omega_1} \approx 2 \left[ 1 - \frac{24h_a}{235\pi T_0} \exp \left( \frac{t}{\tau_{cool}} \right) \right]^{1/2}, \] (33)

Substituting in the expression for $H(t)$ from Eq. (21), to find

\[ \frac{\omega_2}{\omega_1} \approx 2 \left[ 1 - \frac{24h_a}{235\pi T_0} \exp \left( \frac{t}{\tau_{cool}} \right) \right]^{1/2}. \] (34)

It is obvious from this equation that $\omega_2/\omega_1 = P_1/P_2 < 2$ as $t$ increases. Note that the Eqs. (31)-(34) are valid for $h_a/H < 1$. As time increases, the ratio $P_1/P_2$ decreases further from the value of 2. A similar feature is seen in the time independent problems studied, e.g. by Andries et al. (2005a) and Dymova & Ruderman (2006). It was shown by these authors that as the value of the scale height $H$ is decreased, $P_1/P_2$ decreases in value from 2. For the cooling loop, the scale height decreases as $t$ increases, hence we see the period ratio $P_1/P_2$ decrease. Let us recall that magnetic structuring would increase the frequency ratio (Verth & Erdélyi, 2008) so these two physical effects may compete with each other.

Inspired by Verth et al. (2007), Eq. (29) also allows the investigation of the effect of cooling on the eigenfunctions of the transverse oscillations. We are able to obtain an approximate expression for the anti-node shift of the first harmonic of the kink mode. It was shown in Verth et al. (2007) that an exponential density profile can be approximated by the smooth density profile, i.e.

\[ \rho(z, t) = \rho_a \exp \left( \frac{z}{L} \cos \left( \frac{2\pi t}{\tau_{cool}} \right) \right) \approx \rho_a \left[ 1 - \left( 1 - \frac{\rho_a}{\rho_f} \right) \frac{z^2}{L^2} \right]^{-2}, \]

where $\rho_a$ is the density at the apex. The anti-node shift is then approximated as a linear function of $1 - \rho_a/\rho_f$, namely,

\[ \frac{\Delta z_{AN}}{L} \approx 0.04 \left( 1 - \frac{\rho_a}{\rho_f} \right). \] (35)

It can clearly be seen from this approximation that as $t$ increases so does the shift in the anti nodes.

3.2. Numerical Solutions

In order to further progress in solving Eq. (29), we proceed by solving the equation numerically using the shooting method. Eq. (29) is solved for $\rho_a/\rho_e = \chi$ (i.e. $\tau_{cool}$ is the same in both the internal and external plasma) and a number of different, representative values of the cooling time $\tau_{cool}$ that may occur under solar coronal conditions. The values of the cooling time selected reflect the intermediate values of the cooling times that have been reported in EUV observations of coronal loops (Aschwanden & Terradas, 2008), however, the cooling time can be as low as 200 – 300 s (catastrophic cooling) or as large as 34,000 s (see Winebarger et al., 2003). As mentioned, the validity of the WKB method used to solve Eq. (20) becomes increasingly uncertain as $P/\tau_{cool} \rightarrow 1$, so we choose values of $\tau_{cool}$ large enough compared to a fixed representative period, to obtain a reasonable degree of accuracy. As for the opposite end of the spectrum, a large value of $\tau_{cool}$, again, compared to a fixed period, means that any cooling effect will be negligible over the time scale of oscillations. The typical temperature of EUV loops, as detected by e.g. TRACE, is 1.5 MK, we choose this to be the initial value of the temperature, i.e. $T_0$. We also select $h_0$ to be 38 Mm, corresponding to a typical loop length of 120 Mm.

We first plot the change in the value of the period for the fundamental kink mode, Fig. 3. The period is normalised with respect to the value of the period at $t = 0$, $P_i$. It can clearly be seen that the value of the period decreases as time increases and the rate of change of period

\[ \frac{\rho_a}{\rho_f} = \exp \left( -\frac{2L}{\pi H(t)} \right) \approx 1 - \frac{2L}{\pi H(t)}, \] for $L/H(t) \ll 1$. Substituting into Eq. (35) gives

\[ \frac{\Delta z_{AN}}{L} \approx 0.04 \cdot \frac{2L}{\pi H(t = 0)} \left( 1 + \frac{t}{\tau_{cool}} \right). \] (36)

Fig. 6.— Plotted is the normalised amplitude at the loop apex (i.e. the maximum value), $v_{max}/v_{max,t}$, and how amplitude is effected by the loop cooling. The dependence of normalised amplitude is shown for different values of cooling time $\tau_{cool}$ at $T_0 = 1.5$ MK.

Fig. 7.— Shown is the normalised amplitude at the loop apex (i.e. the maximum values). The values of cooling time, $\tau_{cool}$, are in the lower end of the range of values observed in EUV observations.

Now, deviating from the Verth et al. (2007) derivation, we assume that $\rho_a/\rho_f$ is given by a time dependent exponential density profile, in line with

\[ \frac{\rho_a}{\rho_f} = \exp \left( -\frac{2L}{\pi H(t)} \right) \approx 1 - \frac{2L}{\pi H(t)}, \] for $L/H(t) \ll 1$. Substituting into Eq. (35) gives

\[ \frac{\Delta z_{AN}}{L} \approx 0.04 \cdot \frac{2L}{\pi H(t = 0)} \left( 1 + \frac{t}{\tau_{cool}} \right). \] (36)
is dependent on the value of cooling time. If the initial period is $\sim 300$ s, then for $\tau_{cool} = 900$ s the period is seen to decrease by 60% over the typical lifetime of an oscillation ($\sim 1500$ s), giving a period of 120 s. Such a dramatic decrease in period over the typical lifetime of an oscillation has not yet been observed, however this could be simply due to the lack of available high temporal resolution of current EUV imagers onboard satellites. Although it is not shown, it should be noted that the rate of change of the period is also dependent upon value of the initial temperature, $T_0$, and the height of the loop apex in the atmosphere, $h_a$. For $T_0 > 1.5$ MK or $h_a < 38$ Mm the rate of change of the period is less than that seen in Fig. 3. Conversely, for $T_0 < 1.5$ MK or $h_a > 38$ Mm, the rate of change of the period is greater than that seen in Fig. 3.

The period ratio is then obtained for the cooling and plotted against time in Fig. 4. It is obvious from Fig. 4 that as the loop cools the period ratio decreases from its initial value, with the rate of decrease dependent upon the ratio of the period with respect to the cooling time. This result is in line with expectations when considering Eq. (21) and the results from Andries et al. (2005a), Dymova & Ruderman (2006) and Erdélyi & Verth (2007). The decreasing temperature causes a decrease in the value of the scale height $H$, which in turn causes the density along the loop to become increasingly more stratified. The rate at which the period ratio changes is also dependent on the initial temperature and height of the loop apex in the atmosphere.

Let us now analyse how the cooling effects the eigenfunctions of the transverse oscillations. This is possible by finding out how loop cooling influences the amplitude of $v_r$. From Eq. (29) it can be seen that $Q$ is the amplitude of $v_r$, which can then be calculated by solving Eq. (29) numerically. The maximum value of the perturbation has been taken, for the fundamental mode this is the value of the amplitude of the perturbation at the loop apex. The values of the amplitude normalised with respect to the amplitude at $t = 0$, i.e. $v_{max,i}$, are shown in Figs. 5 and 6. It can be seen that as the loop is cooling, the maximum amplitude of the perturbation decreases, i.e. the cooling damps the perturbations. The rate of the damping is dependent upon the initial temperature (Fig. 5) and the ratio of the period with respect to the cooling time scale $\tau_{cool}$ (Fig. 6).

As with the period and the period ratio, the rate of the damping is also dependent upon the height of the loop apex in the atmosphere. A greater value of $h_a$ would mean a greater rate of damping. It appears from this approximate solution that the cooling of the loop could be a clear and dominant source of apparent damping of the perturbations. For $\tau_{cool} > 1000$ s, the damping times for the perturbations shown in Fig. 6 appear to be greater than the time of the observed decay ($\approx 1500$ s, see e.g. Ofman & Aschwanden, 2002). Although not being able to account fully for the observed decay, the cooling would still significantly influence the damping of the oscillations at these timescales.

Let us now investigate the effect of cooling on the
damping for $500 < \tau_{\text{cool}} < 1000$ s. At these typical EUV cooling timescales some caution has to be exercised as to whether the assumption $\delta << 1$ still holds. The initial temperature ($T_0$) is chosen to be typically that of EUV coronal loops, i.e. $\sim 1.5$ MK. It can clearly be seen from Fig. 7 that there is a significant damping in the first 1500 s for $500 < \tau_{\text{cool}} < 700$ s. This indicates that the cooling could have a strong influence on the rate of damping of the loop oscillations.

There is a further interesting effect from cooling on the eigenfunctions. It was shown in Verth et al. (2007) that a longitudinal density profile distorts the eigenfunction of the first harmonic. A strong stratification causes a spatial shift in the anti-nodes of the amplitude profile. The size of the spatial shift was found to increase as the scale height, $H$, decreases. As already stated, decreasing temperature causes a decrease in the value of the scale height $H$, so it should be anticipated that as the temperature decreases there will be a constantly increasing anti-node shift. This is in line with, Eq. (36), the analytic solution for the anti-node shift.

In Fig. 8 it can be seen that this is precisely what occurs. In a magnetic cylinder with length $2L$ with the origin at $L = 0$ and a homogenous density profile, the anti-nodes of the first harmonic occur at $\pm L/2$. The shift caused by the decrease in $H$ causes the anti-nodes to move apart from each other, i.e. each anti-node moves towards the nearest footpoint. The plot in Fig. 8 shows the shift for the anti-node measured from $L/2 + \Delta z_{AN}(t = 0)$ for $T_0 = 1.5$ MK. The anti-node shift at $-L/2 + \Delta z_{AN}(t = 0)$ occurs at the same rate. For $\tau_{\text{cool}} > 900$ s there is only a small shift in the anti-node over the lifetime of a kink oscillation. The anti-node shift is also shown in Fig. 8 for cooling times $< 900$ s. As mentioned previously, caution should be taken due to the possibility of some errors arising from the WKB method. As with all the previous examples (i.e. period, period ratio and damping), the rate of increase of the anti-node shift is dependent upon the initial temperature and the height of the loop apex in the atmosphere.

### 3.3. Testing the theory

It is possible to provide an estimate of the cooling time-scale needed to account for observed damping of the kink oscillations. In Figs. 9-11 a transverse oscillation due to a M 4.6 flare at 12:59:57 on 04 July 1998 observed in a coronal loop is displayed (Aschwanden et al., 2002). The observed coronal loop has a value of $h_a = 56$ Mm when a semi-circular shape is fitted. The oscillation has been overplotted with a fitted damped sine function with the parameter values from Aschwanden et al. (2002). Further to this, the calculated damping due to cooling for a range of $\tau_{\text{cool}}$ is added at different initial temperatures $T_0 = 1.5, 2.0, 2.5$ MK, respectively. It can be seen clearly in Figs. 9-11 that the best fits occur for $T_0 = 1.5$ MK, which is the typical temperature of EUV coronal loops. When $\tau_{\text{cool}} = 700$ s, the calculated damping profile due to cooling is very close to that of the observed profile. It is clearly seen that even if the cooling time is 900 s, the cooling could still have significant influence on the damping of the oscillations.

### 4. CONCLUSION

In this paper we investigated the influence of cooling on the eigenfrequencies and eigenfunctions of transverse MHD oscillations present in a straight, thin magnetic tube embedded in a gravitationally stratified, magnetised atmosphere mimicking coronal EUV loops. The cooling was introduced via a temperature dependent density profile. The ideal MHD equations were then solved to determine the background state and a linear time dependent governing equation was obtained for the perturbations, Eq. (20). An approximate zeroth order solution was obtained for Eq. (20) using the WKB method. From this solution, Eq. (29), it was then possible to establish how the eigenfrequencies and eigenfunctions were influenced, in the leading order, by the loop cooling.

The result, that is of most interest and relevance, is the apparent damping due to cooling. It was shown (Fig. 6) that for oscillations with a typical period of $\sim 300$ s and for cooling timescales between 900 – 2000 s there could be a significant contribution to the damping from the loop cooling. For these cooling timescales, the observed damping may not be completely accounted for by cooling (see, Fig. 9) This may indicate that another mechanism, e.g. resonant absorption (Ruderman & Roberts, 2002; Goossens et al., 2002, Goossens, 2008), may also operate in order to damp even more efficiently the observed transverse oscillations. We also found that for $\tau_{\text{cool}} > 2000$ s the cooling has a less pronounced influence on the damping. Cooling times between 500 – 900 s were also investigated (Fig. 7), however, the results should be treated with some caution as higher order terms from the WKB method may have influence. For 500 – 7000 s it appears that the observed damping could be accounted for almost entirely by the loop cooling. In Figs. 9-11 we confirm this by overplotting a sample observation of a damped transverse oscillation.

Further, the cooling also has a detectable effect on the eigenfunctions of the transverse oscillations. The cooling causes a gradual shift in the spatial position of the anti-nodes of the first harmonic of the kink oscillation. For large cooling times the increase in the anti-node shift over the time-scale of the kink oscillation ($\approx 1500$ s) is small. However, smaller values of the cooling time induce a greater spatial shift of the anti-node over the oscillation time-scale. The rate of change in spatial position of the anti-node is dependent on the cooling time. An analytic
The rate of the change of \( P_1 \) increases from its initial value as the loop cools (Fig. 4). The rate of change of \( P_1 \), also decreases from its initial value as the loop cools (Fig. 4). The rate of the change of \( P_1 \) is obtained upon substitution of Eq. (19) in Eq. (9). The jump condition given in Eq. (12) for the external region is obtained upon substitution of Eq. (19) in Eq. (9). The jump condition given in Eq. (12) implies derivatives of \( v_r \) with respect to \( z \) and \( t \) are continuous across the tube boundary \((r = R)\). The same applies to

\[
\frac{\partial^2 P_T}{\partial t^2} = \nu^2_A(z,t) \frac{\partial^2 P_T}{\partial z^2}.
\]

APPENDIX

A

In this section of the appendix we derive Eq. (6). Differentiating the perturbation of the total magnetic pressure is \( P_T = p_1 + p_m \) with respect to the Lagrangian derivative, \( \partial/\partial t + \mathbf{U} \cdot \nabla \), gives

\[
\frac{\partial P_T}{\partial t} + \mathbf{U} \cdot \nabla P_T = \frac{\partial P_1}{\partial t} + \mathbf{U} \cdot \nabla P_1 + \frac{B_0}{\mu_0} \frac{\partial b_z}{\partial t} + \frac{B_3}{\mu_0} \mathbf{U} \cdot \nabla b_z. \tag{A1}
\]

The derivative of \( b_z \) with respect to time in Eq. (A1) can be eliminated using the \( z \) component of Eq. (5e), Further, substituting in Eq. (5d) into Eq. (A1), differentiating again with respect to the Lagrangian derivative and followed by the substitution of the components of \( v_r \) and \( v_\theta \) from Eq. (5b) leads to

\[
\frac{\partial^2 P_T}{\partial t^2} = -2 \mathbf{U} \cdot \nabla \frac{\partial P_T}{\partial t} - \mathbf{U} \cdot \nabla (\mathbf{U} \cdot \nabla P_T) + \frac{\partial \mathbf{U}}{\partial t} \cdot \nabla P_T + \left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) c_s^2 \left( \frac{\partial p_1}{\partial t} + \mathbf{U} \cdot \nabla p_1 + v_1 \cdot \nabla p_0 \right) + \frac{B_0 \mathbf{U}}{\mu_0} \left( \frac{\partial (rb_z)}{\partial r} + \frac{\partial b_\theta}{\partial \theta} \right) + \frac{B_3}{\mu_0} \left( \frac{\partial (rv_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} \right) + \left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \frac{B_3}{\mu_0} \mathbf{U} \cdot \nabla b_z. \tag{A2}
\]

Using the assumptions made about the flow and the plasma beta made in Section 2.1, Eq. (A2) can then be written

\[
\frac{\partial^2 P_T}{\partial t^2} = \nu^2_A(z,t) \left( \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial P_T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 P_T}{\partial \theta^2} \right) - \frac{B_0 v_A^2(z,t)}{\mu_0} \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial (rb_z)}{\partial r} + \frac{1}{r} \frac{\partial b_\theta}{\partial \theta} \right). \tag{A3}
\]

And finally, using the solenoid equation, (5f), Eq. (A3) reduces to

\[
\frac{\partial^2 P_T}{\partial t^2} = \nu^2_A(z,t) \nu^2 P_T. \tag{A4}
\]

B

In this section the governing equation, Eq. (20), is derived. Substituting Eq. (15) into the equation for the radial velocity perturbation, Eq. (9), an equation valid in the internal region at \( r = R \) is obtained. The analogous equation for the external region is obtained upon substitution of Eq. (19) in Eq. (9). The jump condition given in Eq. (12) implies derivatives of \( v_r \) with respect to \( z \) and \( t \) are continuous across the tube boundary \((r = R)\). The same applies to
and subtract $v^2_{\text{ac}} \times 9$ from $v^2_a \times 9$ internal to obtain

\[
(v^2_{\text{Ac}}(z, t) - v^2_{\text{Ac}}(z, t)) \frac{\partial^2 v_r}{\partial z^2} = \frac{1}{\rho_i - \rho_e} \left( \frac{1}{\rho_i} \frac{\partial v_r}{\partial t} + \frac{1}{\rho_e} \frac{\partial v_r}{\partial t} \right) \frac{\partial v_r}{\partial t} + c_s^2(z, t) b_i \frac{\partial U^2}{\partial z^2} + \frac{1}{\rho_i - \rho_e} \left( \frac{1}{\rho_i} \frac{\partial v_r}{\partial t} - \frac{1}{\rho_e} \frac{\partial v_r}{\partial t} \right)
\]

It is straightforward to obtain from Eqs. (B1) and (B2)

\[
\frac{\partial^2 v_r}{\partial t^2} - \frac{\partial^2 v_r}{\partial z^2} = \frac{1}{\rho_i} \frac{\partial v_r}{\partial t} + \frac{1}{\rho_e} \frac{\partial v_r}{\partial t} \left( \frac{1}{\rho_i} \frac{\partial v_r}{\partial t} - \frac{1}{\rho_e} \frac{\partial v_r}{\partial t} \right) + c_s^2(z, t) b_i \frac{\partial U^2}{\partial z^2} + \frac{1}{\rho_i - \rho_e} \left( \frac{1}{\rho_i} \frac{\partial v_r}{\partial t} - \frac{1}{\rho_e} \frac{\partial v_r}{\partial t} \right)
\]

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